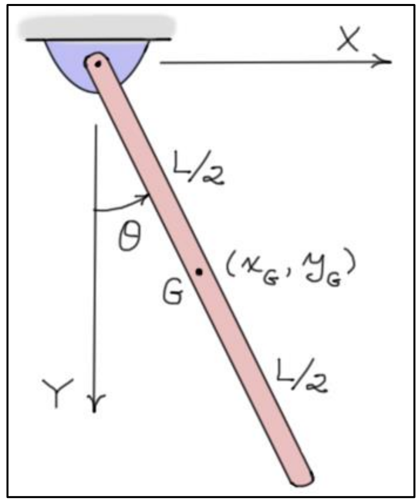
Intermediate Dynamics
Configuration Constraints for Mechanical Systems



Suppose the ***configuration*** of a mechanical system is defined by "n" generalized coordinates, say $q_k \; (k=1,\ldots,n)$. These coordinates may all be ***independent***, or they may form a ***dependent*** set. For example, consider the simple pendulum shown in the diagram. The coordinate set $\{x_G, y_G, \theta\}$ is a ***dependent*** coordinate set. The following equations can be used to relate the coordinates.

$$\boxed{x_G = \tfrac{L}{2}\sin(\theta)} \qquad \boxed{y_G = \tfrac{L}{2}\cos(\theta)}.$$

Hence, for this system only ***one*** generalized coordinate is required to define its position. Any set of ***two*** or ***more*** coordinates forms a ***dependent*** set.

The types of constraints described above are referred to as ***configuration constraints***. In general, for a mechanical system described by "n" generalized coordinates $q_k \; (k=1,\ldots,n)$ with "m" configuration constraints, the constraints can be written as

$$\boxed{f_j(q_1, q_2, \ldots, q_n, t) = 0} \qquad (j=1,\ldots,m)$$

Here, the constraints may also be dependent on the time, t. Configuration constraints are most useful in a dynamic analysis when they are differentiated into a form that is ***linear*** in the time derivatives of the coordinates.

$$\boxed{\frac{df_j}{dt} = \sum_{k=1}^{n}\left(\frac{\partial f_j}{\partial q_k}\right)\dot{q}_k + \frac{\partial f_j}{\partial t} = 0} \qquad (j=1,\ldots,m)$$

or

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m)$$

Examples

1. For the simple pendulum shown above, define the generalized coordinate set as $\{q_1, q_2, q_3\} = \{x_G, y_G, \theta\}$. Then, the two configuration constraints are as follows.

$$\boxed{\begin{Bmatrix} f_1(q_1,q_2,q_3) \\ f_2(q_1,q_2,q_3) \end{Bmatrix} = \begin{Bmatrix} q_1 - \frac{L}{2}\sin(q_3) \\ q_2 - \frac{L}{2}\cos(q_3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Differentiating these equations gives

$$\begin{Bmatrix} \dot{q}_1 - \frac{L}{2}\dot{q}_3\cos(q_3) \\ \dot{q}_2 + \frac{L}{2}\dot{q}_3\sin(q_3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \boxed{\begin{bmatrix} 1 & 0 & -\frac{L}{2}\cos(q_3) \\ 0 & 1 & +\frac{L}{2}\sin(q_3) \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Comparing these results to the general form shown above, we conclude

$$\boxed{a_{11} = a_{22} = 1} \qquad \boxed{a_{12} = a_{21} = 0} \qquad \boxed{a_{13} = -\frac{L}{2}\cos(\theta)} \qquad \boxed{a_{23} = +\frac{L}{2}\sin(\theta)}$$

and

$$\boxed{a_{j0} = 0} \qquad (j = 1,2)$$

2. For the simple pendulum shown above, define the generalized coordinate set as $\{q_1, q_2\} = \{x_G, y_G\}$. Then, the configuration constraint is

$$\boxed{f(q_1,q_2) = q_1^2 + q_2^2 - (L/2)^2 = 0}$$

Differentiating the constraint gives

$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 = 0 \qquad \text{or} \qquad \boxed{q_1\dot{q}_1 + q_2\dot{q}_2 = 0}$$

So, in this case, $\boxed{a_{11} = q_1 = x_G}$, $\boxed{a_{12} = q_2 = y_G}$, and $\boxed{a_{10} = 0}$.