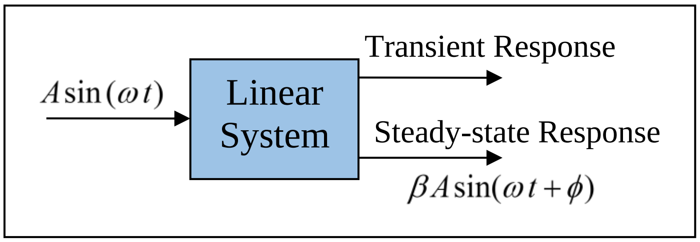
Introductory Control Systems
Introduction to Bode Diagrams

The ***frequency response*** of a linear system is defined as the ***steady-state response*** to a ***sinusoidal (harmonic) input***. If the ***input*** to a linear system is ***sinusoidal***, then the ***steady-state output*** will also be sinusoidal. The output ***differs*** from the ***input*** in ***magnitude*** and ***phase***, ***only***. One way of describing this harmonic, steady-state response is a Bode diagram.



To find the ***magnitude*** and ***phase*** of the ***steady state response*** of a system with transfer function $T(s)$, replace "s" with "$j\omega$" in the transfer function, and then evaluate the magnitude and phase of $T(j\omega)$.

$$T(s) \quad \rightarrow \quad \boxed{T(j\omega)=M(\omega)e^{j\phi(\omega)}}$$

The results are presented by plotting: 1) $M(\omega)$ in decibels (dB) vs. $\log_{10}(\omega)$, and 2) $\phi(\omega)$ in radians (or degrees) vs. $\log_{10}(\omega)$. These two plots form the ***Bode diagram***.

Review of Products and Ratios of Complex Numbers

Given ***two complex numbers*** $A=ae^{j\alpha}$ and $B=be^{j\beta}$ (expressed in polar form), the product and ratio of the two numbers can be written as

$$\boxed{AB=abe^{j(\alpha+\beta)}} \qquad \text{and} \qquad \boxed{A/B=(a/b)e^{j(\alpha-\beta)}}$$

The magnitudes of the product and ratio can be expressed as a "***raw***" magnitude or a magnitude in ***decibels*** (dB).

Raw Magnitude	Magnitude in dB
$\lvert AB\rvert=ab$	$\lvert AB\rvert_{dB}=20\log(ab)=20\log(a)+20\log(b)$
$\lvert A/B\rvert=a/b$	$\lvert AB\rvert_{dB}=20\log(a/b)=20\log(a)-20\log(b)$

When the magnitude is expressed in dB, both the magnitude and the phase of the individual components of the transfer function are ***additive***.

Magnitude and Phase of a Transfer Function

As an example, consider a linear system with the transfer function

$$\boxed{T(s)=\frac{K(s+a)}{(s+b)(s^2+2\zeta\omega_n s+\omega_n^2)}}$$

To find $M(\omega)$ and $\phi(\omega)$ the ***magnitude*** and ***phase*** as a function of frequency, substitute $s = j\omega$ into the transfer function to get

$$\begin{aligned}T(j\omega)&=\frac{K(j\omega+a)}{(j\omega+b)((j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2)}\\&=\frac{K(a+j\omega)}{(b+j\omega)\left(\left(\omega_n^2-\omega^2\right)+j2\zeta\omega_n\omega\right)}\end{aligned}$$

The magnitude and phase of the system can then be written as the sum of the magnitudes and phases of the individual terms.

$$\boxed{M(\omega)=20\log\left|K\right|+20\log\left|a+j\omega\right|-20\log\left|b+j\omega\right|-20\log\left|(\omega_n^2-\omega^2)+j2\zeta\omega_n\omega\right|}$$

$$\boxed{\phi(\omega)=\angle K+\angle\left(a+j\omega\right)-\angle\left(b+j\omega\right)-\angle\left((\omega_n^2-\omega^2)+j2\zeta\omega_n\omega\right)}$$

Note that terms in the ***denominator*** of the transfer function are ***negated*** in the sum. Note also that the "***log***" function can be ***positive*** or ***negative***, depending on the size of the argument. So, ***magnitude terms*** in both the ***numerator*** and ***denominator*** can be either ***positive*** or ***negative***.

Magnitude and Phase of Individual Terms

To get a better understanding of the frequency response of the individual terms in the transfer function, the frequency responses of common terms are approximated below using ***asymptotes***.

Gain: (K)

$$M(\omega)=+20\log\left|K\right|$$

$$\phi(\omega)=\begin{cases}0\text{ (deg)} & (K>0)\\ 180\text{ (deg)} & (K<0)\end{cases}$$

The magnitude and phase of the gain is the ***same*** at all frequencies. For $K>1$ this term represents a ***perfect amplifier***.

Zero at the origin of order N: (s^N)

$M(\omega) = +20\log(\omega^N) = +20N\log(\omega)$ … This is a line with $+20N\,(dB/\text{decade})$ slope.

$\varphi(\omega) = +90N$ (deg)

The magnitude of the response of a zero at the origin ***increases*** as a function of frequency, and its phase is a ***positive constant***. For $N=1$, this term represents a ***perfect differentiator*** – it provides ***phase advance***, but it can become ***too responsive*** at high frequencies.

Pole at the origin of order N: $(1/s^N)$

$M(\omega) = -20\log(\omega^N) = -20N\log(\omega)$ … This is a line with $-20N\,(dB/\text{decade})$ slope.

$\varphi(\omega) = -90N$ (deg)

The magnitude of the response of a pole at the origin ***decreases*** as a function of frequency, and its phase is a ***negative constant***. For $N=1$, this term represents a ***perfect integrator*** – it is more responsive at lower frequencies, but it suffers from ***phase lag***.

Real Zero $(s+a)$: (in the left-half plane)

Corner frequency at a (radians/sec)

$M(\omega) = +20\log|a + j\omega| = +20\log\left(\sqrt{a^2+\omega^2}\right)$

$\phi(\omega) = \tan^{-1}(\omega/a)$

Low frequency asymptote: $(\omega \ll a)$

$M(\omega) \approx +20\log|a| = \text{constant}$

$\phi(\omega) \approx 0$ (deg)

High frequency asymptote: $(\omega \gg a)$

$M(\omega) \approx +20\log|\omega|$ … This is a line with $+20\,(dB/\text{decade})$ slope.

$\phi(\omega) \approx 90$ (deg)

A ***real zero*** has a ***nearly constant*** magnitude response at frequencies ***well below*** its ***corner frequency***, and its response ***increases*** at $20\,(dB/\text{decade})$ at frequencies above the corner. At frequencies ***below*** the ***corner***, it has ***zero phase***, and it offers ***phase advance*** at frequencies ***above*** the ***corner***. It can become ***too responsive*** at frequencies ***well above*** the corner.

Real Pole $\left(1/\left(s+b\right)\right)$: (in the left-half plane)

Corner frequency at b (radians/sec)

$M(\omega) = -20\log\left|b + j\omega\right| = -20\log\left(\sqrt{b^2+\omega^2}\right)$

$\phi(\omega) = -\tan^{-1}\left(\omega/b\right)$

Low frequency asymptote: $\left(\omega \ll b\right)$

$M(\omega) \approx -20\log\left|b\right| = \text{constant}$

$\phi(\omega) \approx 0$ (deg)

High frequency asymptote: $\left(\omega \gg b\right)$

$M(\omega) \approx -20\log\left|\omega\right|$ … This is a line with $-20\ (dB/\text{decade})$ slope.

$\phi(\omega) \approx -90$ (deg)

A ***real pole*** has a ***nearly constant*** magnitude response at frequencies ***well below*** its ***corner frequency***, and its response ***decreases*** at $20\ (dB/\text{decade})$ at frequencies above the corner. At frequencies ***below*** the ***corner***, it has ***zero phase***, and it has ***phase lag*** at frequencies ***above*** the ***corner***.

Complex Poles $1/\left(s^2+2\zeta\omega_n s+\omega_n^2\right)$: (in the left-half plane)

Corner frequency at ω_n (radians/sec)

$M(\omega) = -20\log\left|(\omega_n^2-\omega^2) + j2\zeta\omega\omega_n\right|$

$\phi(\omega) = -\tan^{-1}\left[2\zeta\omega\omega_n/\left(\omega_n^2-\omega^2\right)\right]$

Low frequency asymptote: $\left(\omega \ll \omega_n\right)$

$M(\omega) \approx -20\log\left(\omega_n^2\right) = -40\log(\omega_n) = \text{constant}$

$\phi(\omega) \approx 0$ (deg)

High frequency asymptote: $\left(\omega \gg \omega_n\right)$

$M(\omega) \approx -20\log\left(\omega^2\right) = -40\log(\omega)$ … This is a line with $-40\ (dB/\text{decade})$ slope.

$\phi(\omega) \approx -180$ (deg)

A ***pair*** of ***complex poles*** has a ***nearly constant*** magnitude response at frequencies ***well below*** its ***corner frequency***, and its response ***decreases*** at $40\ (dB/\text{decade})$ at frequencies above the corner (twice the rate of a single pole). At frequencies ***below*** the ***corner***, it has ***zero phase***, and it has ***phase lag*** at frequencies ***above*** the ***corner***.

Construction of Approximate Bode Magnitude Diagrams

The use of ***logarithms*** allows the ***magnitudes*** and ***phases*** of different terms in a transfer function of a system to be ***additive***. The magnitude of the transfer function generally varies somewhat smoothly with frequency, so ***asymptotes*** can be used to generate simple approximations to the magnitude diagram. Unfortunately, the phase transitions with frequency are not as easily represented.

For example, the magnitude response of a ***second order system*** is well represented by asymptotes ***except*** near the system's ***natural frequency***. The approximation at the natural frequency gets worse as the damping ratio is lowered. The phase approximation is good for low damping ratios, but it is only good at frequencies well below and well above the natural frequency for higher damping ratios.

Higher order systems fall into ***two general categories*** – system's that have a ***pole*** or ***zero*** at the ***origin*** and those that ***do not***. The construction of approximate magnitude diagrams for each of these two categories is described below. Phase diagrams of these systems are not easily approximated and will not be considered here.

Systems that ***do not have*** a ***pole*** or ***zero*** at the ***origin*** have a ***constant magnitude response*** at ***low frequencies***, that is, at frequencies well below the system's lowest corner frequency. This constant magnitude is the sum of the constant magnitudes of all the terms in the transfer function. This magnitude is held until the first ***corner frequency*** is passed. At the crossing of each corner frequency, the slope of the diagram is ***changed*** based on the type of term whose corner is passed. For example, a ***real zero increases*** the slope by 20 (dB/decade), while a ***pair*** of ***complex poles decreases*** the slope by 40 (dB/decade). These are ***changes*** in ***slope***, not absolute slopes.

Systems that ***have*** a ***pole*** or ***zero*** at the ***origin*** are ***not constant*** below the system's lowest corner frequency. The ***slope*** of the diagram at these frequencies is the slope ***associated*** with the pole or zero at the origin. For example, if a system has a ***pole*** of ***order one*** at the ***origin***, then its slope at low frequencies will be -20 (dB/decade). If a system has a ***zero*** of ***order one*** at the ***origin***, then its slope at low frequencies will be $+20$ (dB/decade). For these systems, the magnitude should be ***calculated*** at some frequency ***below*** the ***lowest corner***, and a line with the appropriate low-frequency slope should be drawn through that point. That slope is maintained until the next corner frequency is crossed. As noted above, the slope changes as each corner is crossed.

For further clarification, consider ***three systems*** whose ***first corner frequency*** is associated with a ***real zero*** in the transfer function. The ***first system*** has ***no pole*** or ***zero*** at the ***origin***, the ***second*** has a ***pole*** of ***order one*** at the ***origin***, and the ***third*** has a ***zero*** of ***order one*** at the ***origin***. The magnitude plot of the ***first system starts*** with a ***zero slope*** that ***increases*** to a $+20$ (dB/decade) slope when it crosses the corner. The ***second system starts*** with a slope of -20 (dB/decade) and ***increases*** to a ***zero slope*** when it crosses the corner. The ***third system starts*** with a slope of $+20$ (dB/decade) which ***increases*** to $+40$ (dB/decade) when the corner is crossed.

The ***concepts*** presented above can be used to ***construct approximate Bode magnitude diagrams***. The ***motivation*** for constructing these diagrams is for the analyst to develop an ***understanding*** of how ***individual terms*** in the system transfer function ***affect the overall system response***. This understanding enables the analyst to ***introduce terms*** into the system transfer function using a ***compensator*** (or controller) that will have a ***positive influence*** on the response. Unfortunately, a meaningful approximate Bode phase diagram is not as easy to construct, so computer algorithms will generally be used to aid in detailed compensator design.