Introductory Motion and Control
Difference Equations and Approximate Solutions to Ordinary Differential Equations

Background

- Numerical methods can be used to ***convert ordinary differential equations (ODE's)*** into ***difference equations***. These difference equations can be used to generate ***approximate solutions*** to the original ODE.
- ***Many numerical methods exist*** for this purpose, and generally they may be used to solve ***linear*** or ***non-linear*** ODE's. The solutions are a series of (discrete) values that approximate the solution.
- The ***accuracy*** of the approximate solution depends on the ***method*** used and the ***increment*** between successive values.
- In the notes that follow, it is assumed that the ***independent variable*** is ***time*** t, and that the sequence of values that represent the approximate solution will be ***equally spaced*** in time. The time increment between successive values is represented by T.
- The sequence of $N+1$ time values at which the solution is approximated can then be written as $t=\left(0,T,2T,\ldots,kT,\ldots,NT\right)$.

Euler's Method: ***Single, First-Order ODE***

- Consider a first order ODE given by the equation

$$\dot{x}=f(x,t) \tag{1}$$

- Here x is the unknown variable, and f is a function of the unknown variable and time. To find an approximate solution to the differential equation, we expand the unknown $x(t)$ in a ***Taylor series*** about an arbitrary time $t=kT$.
- In ***Euler's method***, the terms of order T^2 and higher are assumed to be small and are ignored.

$$\begin{aligned} x\left((k+1)T\right) &= x(kT)+T\left.\frac{dx}{dt}\right|_{t=kT}+\tfrac{1}{2}T^2\left.\frac{d^2x}{dt^2}\right|_{t=kT}+\tfrac{1}{3!}T^3\left.\frac{d^3x}{dt^3}\right|_{t=kT}+\cdots \\ &\approx x(kT)+T\left.\frac{dx}{dt}\right|_{t=kT} \end{aligned} \tag{2}$$

- Using the differential equation to calculate the derivative of x in the truncated Taylor series gives the ***difference*** equation

$$\boxed{x\left((k+1)T\right)=x(kT)+\left(T\cdot f\left(x(kT),kT\right)\right)} \tag{3}$$

- Eq. (3) can now be used to produce a sequence of values that approximate the solution to Eq. (1). The solution is second order accurate over a single step; however, it is only ***first-order accurate*** over ***many steps***. Consequently, very small time steps are often required to produce reasonably accurate results.

- It should be noted that Eq. (3) predicts the value of x at time $t=(k+1)T$ based on the value of x and its derivative $\dot{x}$ at time $t=kT$. This is based on the idea that $\dot{x}$ at time $t=kT$ can be approximated using a ***first-order***, ***forward difference***. That is,

$$\left.\frac{dx}{dt}\right|_{t=kT} \approx \frac{x\big((k+1)T\big)-x(kT)}{T} \tag{4}$$

- A ***modified Euler method*** has been developed that predicts the value of x at time $t=(k+1)T$ based on the value of x at time $t=kT$ and the ***average*** of $\dot{x}$ at times $t=kT$ and $t=(k+1)T$. Since the value of $\dot{x}$ at time $t=(k+1)T$ is not known, it is also estimated as part of the method. The details are not included here.

<u>Runge-Kutta Method</u>: ***Single, First-Order ODE***

- ***One*** of the ***most popular methods*** for generating an approximate solution to Eq (1) is the Runge-Kutta method. Runge-Kutta methods of various levels of accuracy have been developed.

- The ***second-order*** Runge-Kutta method uses the following ***difference*** equations

$$x\big((k+1)T\big)=x(kT)+a\Delta_1+b\Delta_2 \tag{5}$$

where

$$\begin{aligned}\Delta_1 &= T\cdot f(x(kT),kT)\\ \Delta_2 &= T\cdot f(x(kT)+\beta\Delta_1,kT+\alpha T)\end{aligned} \tag{6}$$

- a, b, α, and β are ***all constants***. Equating this expansion to a Taylor series expansion through the T^2 terms gives the following three equations that must be satisfied by the four unknown constants.

$$a+b=1 \qquad \alpha b=\tfrac{1}{2} \qquad \beta b=\tfrac{1}{2} \tag{7}$$

- Using the values $a=b=\frac{1}{2}$ and $\alpha=\beta=1$ provides the ***same equations used for the modified Euler method*** mentioned above. This method is third-order accurate for a single step and ***second-order accurate for many steps***.

- ***One*** of the ***most popular methods*** used is a ***fourth order, Runge-Kutta method***. A commonly used set of ***difference*** equations used for this method are

$$x\big((k+1)T\big) = x(kT) + \tfrac{1}{6}\big(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\big) \tag{8}$$

where

$$\begin{aligned}
\Delta_1 &= T \cdot f\,(x(kT), kT) \\
\Delta_2 &= T \cdot f\,(x(kT) + \tfrac{1}{2}\Delta_1, kT + \tfrac{1}{2}T) \\
\Delta_3 &= T \cdot f\,(x(kT) + \tfrac{1}{2}\Delta_2, kT + \tfrac{1}{2}T) \\
\Delta_4 &= T \cdot f\,(x(kT) + \Delta_3, kT + T)
\end{aligned} \tag{9}$$

This method is fifth-order accurate for one step but only ***fourth-order accurate*** for ***many steps***.

<u>Multiple, First-Order ODE's</u>

- Consider a set of m first-order, ODE's given by the equation

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}\,(\underset{\sim}{x}, t) \tag{10}$$

- Here $\underset{\sim}{x}$ represents an $m \times 1$ vector of unknown variables and $\underset{\sim}{f}$ represents an $m \times 1$ vector of functions of the unknowns and time. An ***approximate solution*** for each of the unknowns may be found by expanding each of the m unknowns $x_i(t)$ in a Taylor series about an arbitrary time $t = kT$.
- Using the results presented in Eq. (3), the difference equations for a ***first-order, Euler approximations*** are

$$\boxed{x_i\big((k+1)T\big) = x_i(kT) + \big(T \cdot f_i(\underset{\sim}{x}(kT), kT)\big)} \qquad \big(i = 1, \ldots, m\big). \tag{11}$$

- Using the results presented in Equations (8) and (9), the difference equations for ***fourth-order, Runge-Kutta approximations*** are

$$\boxed{x_i\big((k+1)T\big) = x_i(kT) + \tfrac{1}{6}\big(\Delta_{i1} + 2\Delta_{i2} + 2\Delta_{i3} + \Delta_{i4}\big)} \tag{12}$$

where

$$\boxed{\begin{aligned}
\Delta_{i1} &= T \cdot f_i(\underset{\sim}{x}(kT), kT) \\
\Delta_{i2} &= T \cdot f_i(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_1, kT + \tfrac{1}{2}T) \\
\Delta_{i3} &= T \cdot f_i(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_2, kT + \tfrac{1}{2}T) \\
\Delta_{i4} &= T \cdot f_i(\underset{\sim}{x}(kT) + \underset{\sim}{\Delta}_3, kT + T)
\end{aligned}} \qquad \big(i = 1, \ldots, m\big) \tag{13}$$

Here, $\underset{\sim}{\Delta}_j \ (j = 1,2,3)$ are $m \times 1$ vectors whose elements are $\Delta_{ij} \ \ (i = 1, \ldots, m)$.

Example 1: $\boxed{\frac{dx}{dt} + ax = u(t)}$ with $x(0) = 0$ and $a = \text{constant}$. (***1st order, linear ODE***)

Euler's Method

$$x\big((k+1)t\big) = x(kT) + T \cdot f\big(x(kT), kT\big)$$

or

$$\boxed{x\big((k+1)T\big) = x(kT) + T\big[u(kT) - ax(kT)\big] = (1 - aT)x(kT) + Tu(kT)} \qquad (14)$$

Fourth Order, Runge-Kutta

$$\boxed{x\big((k+1)T\big) = x(kT) + \tfrac{1}{6}\big(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4\big)} \qquad (15)$$

where

$$\boxed{\begin{aligned} \Delta_1 &= T\big[u(kT) - ax(kT)\big] \\ \Delta_2 &= T\Big[u(kT + \tfrac{1}{2}T) - a\big(x(kT) + \tfrac{1}{2}\Delta_1\big)\Big] \\ \Delta_3 &= T\Big[u(kT + \tfrac{1}{2}T) - a\big(x(kT) + \tfrac{1}{2}\Delta_2\big)\Big] \\ \Delta_4 &= T\Big[u\big((k+1)T\big) - a\big(x(kT) + \Delta_3\big)\Big] \end{aligned}} \qquad (16)$$

Results

- Letting the parameter $a = 5$, the ***input*** $u(t) = u_s(t)$ (the ***unit step*** function), and the time step $T = 0.1\ (\text{sec})$, the two methods provide the results shown in ***Fig 1***.
- Note that even with a fairly large time step, the ***Runge-Kutta*** approximation ***matches*** the exact solution very closely.

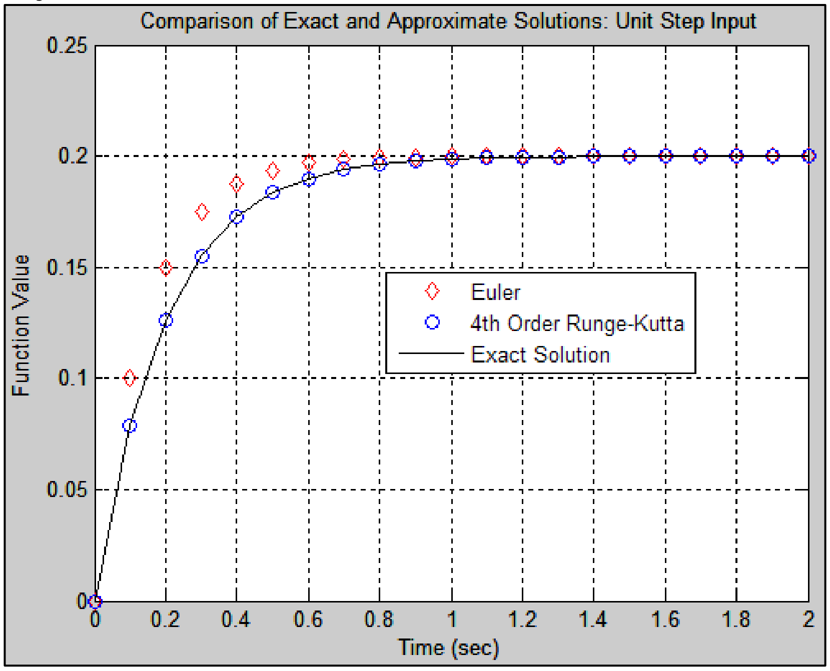

Fig 1. Comparison of Exact and Approximate Results – Unit Step Input

- Letting the parameter $a = 5$, the ***input*** $u(t) = 32e^{-3t}\cos(2t)$, and the time step $T = 0.02$ (sec), the two methods provide the results shown in ***Fig 2***.
- With a more ***active input***, a smaller time step is required to generate accurate results. Both methods provide reasonably accurate results; however, the Euler approximation over-predicts the peak response while the Runge-Kutta method under-predicts it.

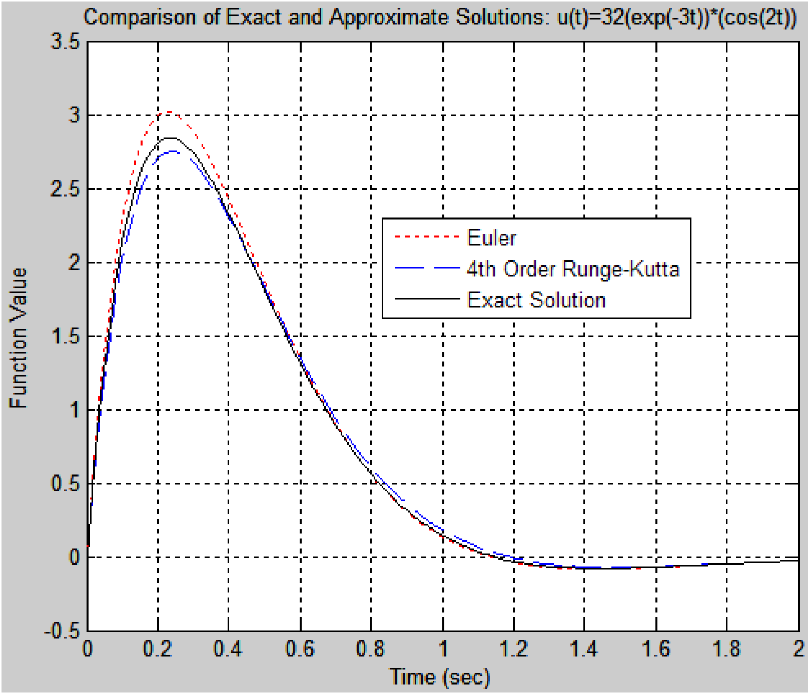

Fig 2. Comparison of Exact and Approximate Results: $u(t) = 32e^{-3t}\cos(2t)$

<u>Example 2</u>: $\boxed{\dfrac{d^2\theta}{dt^2} + \left(\dfrac{g}{\ell}\right)\sin(\theta) = 0}$ with $\theta(0) = \pi/4$ and $\dot{\theta}(0) = 0$. (***2nd order, non-linear***)

- To solve a second-order ODE, the equation must first be written as two first-order ODE's. To do this, let $x = \theta$ and $y = \dot{\theta} = \dot{x}$. Using the variables x and y, the original ODE can be written as

$$\boxed{\begin{aligned} \dot{x} &= y = f_1(x,y,t) \\ \dot{y} &= -(g/\ell)\sin(x) = f_2(x,y,t) \end{aligned}} \quad \text{with } x(0) = \pi/4 \text{ and } y(0) = 0 \qquad (17)$$

<u>Euler's Method</u>

$$x\big((k+1)T\big) = x(kT) + T \cdot f_1\big(x(kT), y(kT), kT\big)$$
$$y\big((k+1)T\big) = y(kT) + T \cdot f_2\big(x(kT), y(kT), kT\big)$$

$$\Rightarrow \boxed{\begin{aligned} x\big((k+1)T\big) &= x(kT) + T\,y(kT) \\ y\big((k+1)T\big) &= y(kT) - T\,(g/\ell)\sin\big(x(kT)\big) \end{aligned}} \qquad (18)$$

Fourth Order, Runge-Kutta

$$
\boxed{\begin{aligned} x\big((k+1)T\big) &= x(kT) + \tfrac{1}{6}\big(\Delta_{11} + 2\Delta_{12} + 2\Delta_{13} + \Delta_{14}\big) \\ y\big((k+1)T\big) &= y(kT) + \tfrac{1}{6}\big(\Delta_{21} + 2\Delta_{22} + 2\Delta_{23} + \Delta_{24}\big) \end{aligned}} \tag{19}
$$

where

$$
\boxed{\begin{aligned}
\Delta_{11} &= T \cdot f_1(\underset{\sim}{x}(kT), kT) = T\, y(kT) \\
\Delta_{21} &= T \cdot f_2(\underset{\sim}{x}(kT), kT) = -T\left(g/\ell\right)\sin\left(x(kT)\right) \\
\Delta_{12} &= T \cdot f_1(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_1, kT + \tfrac{1}{2}T) = T\left(y(kT) + \tfrac{1}{2}\Delta_{21}\right) \\
\Delta_{22} &= T \cdot f_2(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_1, kT + \tfrac{1}{2}T) = -T\left(g/\ell\right)\sin\left(x(kT) + \tfrac{1}{2}\Delta_{11}\right) \\
\Delta_{13} &= T \cdot f_1(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_2, kT + \tfrac{1}{2}T) = T\left(y(kT) + \tfrac{1}{2}\Delta_{22}\right) \\
\Delta_{23} &= T \cdot f_2(\underset{\sim}{x}(kT) + \tfrac{1}{2}\underset{\sim}{\Delta}_2, kT + \tfrac{1}{2}T) = -T\left(g/\ell\right)\sin\left(x(kT) + \tfrac{1}{2}\Delta_{12}\right) \\
\Delta_{14} &= T \cdot f_1(\underset{\sim}{x}(kT) + \underset{\sim}{\Delta}_3, kT + T) = T\left(y(kT) + \Delta_{23}\right) \\
\Delta_{24} &= T \cdot f_2(\underset{\sim}{x}(kT) + \underset{\sim}{\Delta}_3, kT + T) = -T\left(g/\ell\right)\sin\left(x(kT) + \Delta_{13}\right)
\end{aligned}} \tag{20}
$$

Results

- Letting $g = 9.81\,(\text{m}/s^2)$, the length $\ell = 0.5\,(\text{m})$, and the time step $T = 0.01\,(\text{sec})$, the two methods produce the results shown in ***Fig 3*** (next page) for the angle of the pendulum. Since no damping was included, the ***pendulum*** should ***continue to oscillate*** between $\pm\dfrac{\pi}{4} = \pm 0.7854\ (\text{rad})$.
- Clearly, the ***Euler method is unstable*** at this time step. Conversely, the ***Runge-Kutta method*** appears to be ***tracking*** the solution ***very well***.
- ***Fig 4*** (next page) shows results for the two methods for a ***smaller time step*** $T = 0.001\,(\text{sec})$. The ***Euler method*** is producing much better results at this time step. However, after four periods, it is evident that the oscillations are beginning to increase in amplitude, indicating that the method ***may*** still be ***unstable***.

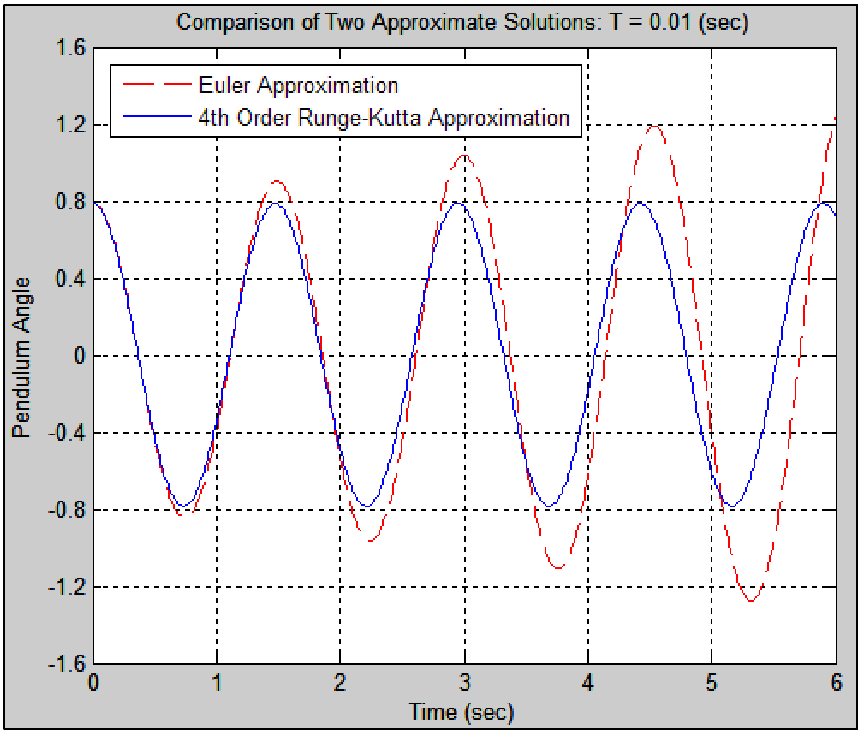

Figure 3.
Euler & Runge-Kutta Approximations to Pendulum Equation (T=0.01 (sec))

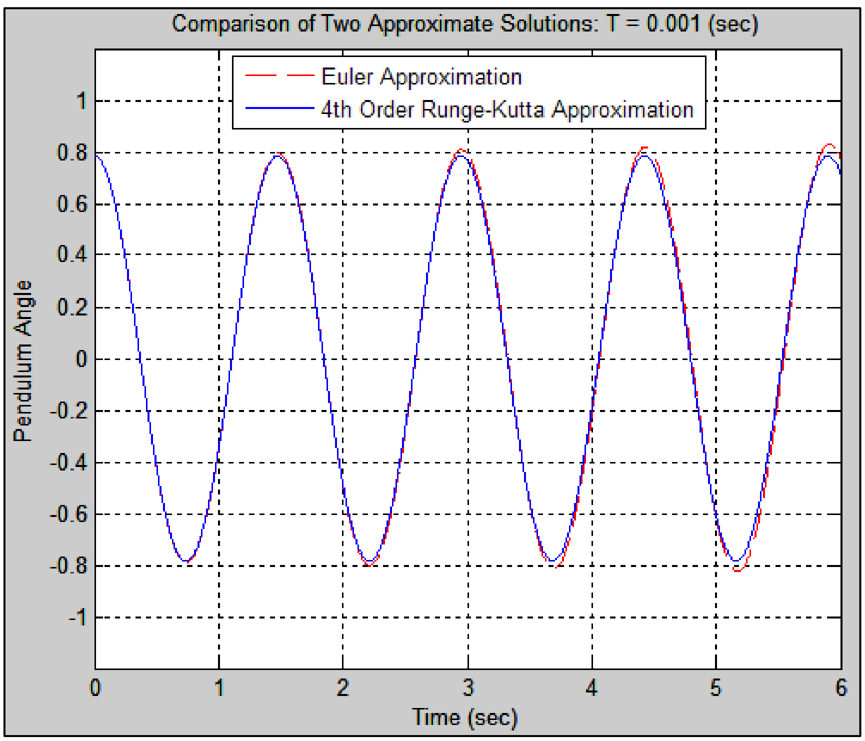

Figure 4.
Euler & Runge-Kutta Approximations to Pendulum Equation (T=0.001 (sec))