Introductory Control Systems
PID Control of a Spring-Mass-Damper (SMD) Position

Fig. 1 shows a ***spring-mass-damper*** system with a ***force actuator*** for ***position control***. The spring has stiffness k, the damper has coefficient c, the block has mass m, and the position of the mass is measured by the variable x. As discussed in earlier notes, the ***transfer function*** of the SMD with the actuating force F_a as input and the position x as output is

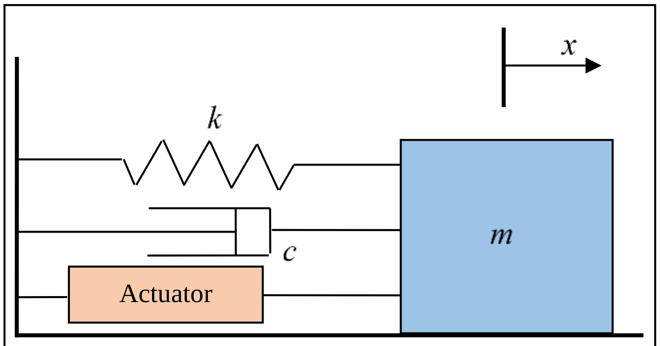
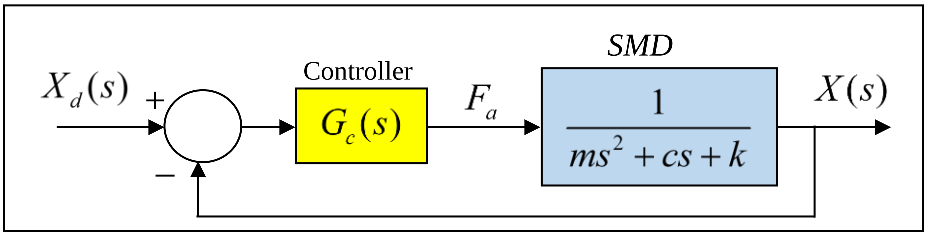


Figure 1. Spring-Mass-Damper System with Force Actuator

$$\boxed{\frac{X}{F_a}(s) = \frac{1}{ms^2 + cs + k}} \qquad (1)$$

Assuming ***ideal actuator*** and ***sensor*** responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, X_d represents the ***desired position***, and $G_c(s)$ represents the ***transfer function*** of the controller.



In the following analyses, the SMD parameters are assumed to be: $m = 1$ slug, $c = 8.8$ (lb-s/ft), and $k = 40$ (lb/ft). This represents an ***under-damped***, ***second-order*** plant with

$$\omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)} \ \ldots \text{ natural frequency}$$
$$\zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7 \ \ldots \text{ damping ratio}$$

Proportional Control

If simple ***proportional control*** is used, then $\boxed{G_c(s) = K}$. In this case, the loop transfer function and closed-loop transfer functions are

$$\boxed{GH(s) = \frac{K}{s^2 + 8.8s + 40}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K}{s^2 + 8.8s + (40 + K)}} \qquad (2)$$

This is a ***type-zero*** system and hence will have a ***finite steady-state error*** for a step input. Using the ***final-value theorem*** and the ***closed-loop transfer function***, x_{ss} the final value of $x(t)$ to a unit step command is

$$\boxed{x_{ss} = \lim_{s \to 0}\left(s \cdot \frac{1}{s} \cdot \frac{K}{s^2 + 8.8s + (40+K)} \right) = \frac{K}{40+K} < 1} \qquad (3)$$

Eq. (3) indicates that ***large values of*** K ***lead to small steady-state error***; however, as seen below, they also lead to a ***faster, less damped responses***.

The root locus diagram for the closed-loop system for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 2***. Note that as the value of K is ***increased***, the closed-loop poles move straight up/down, indicating that the natural frequency is ***increased,*** and the damping ratio is ***decreased***. Also, as the value of K is ***increased***, the ***phase (stability) margin is decreased.***

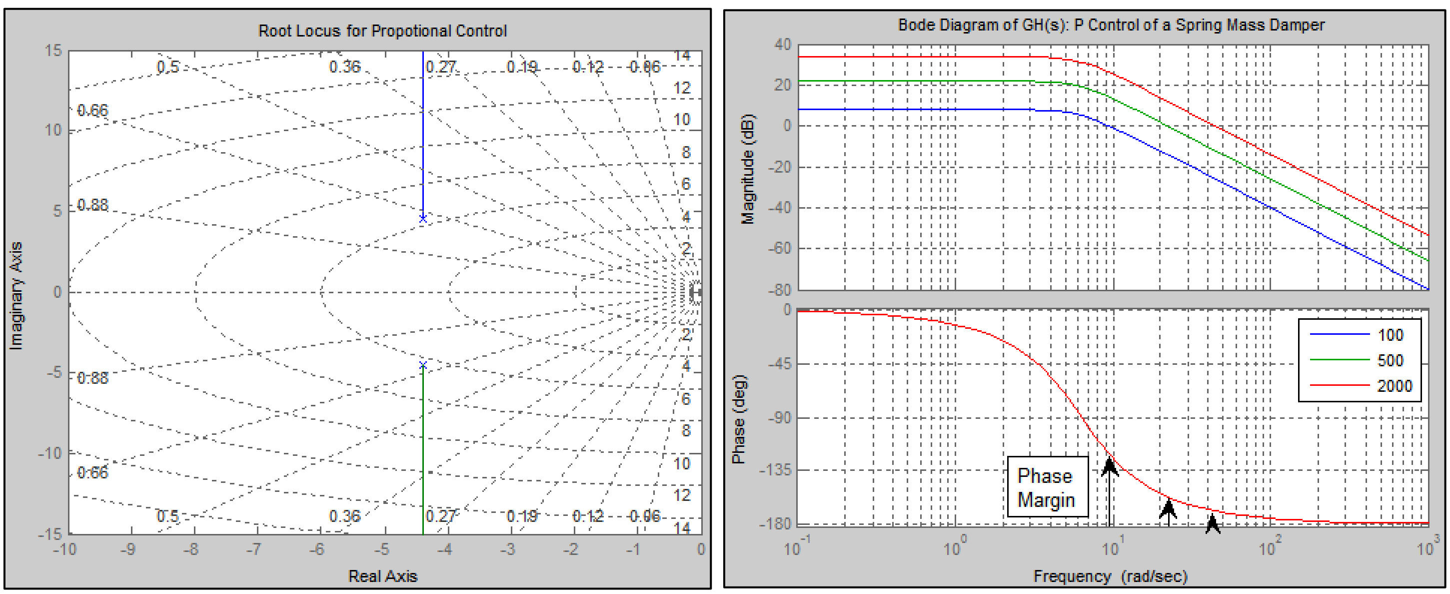


Figure 2. Root Locus Diagram and Bode Diagram for ($GH(s)$) for Proportional Control

Fig. 3 shows ***step responses*** and ***Bode diagrams*** of the closed-loop system for proportional gains K of 100, 500, and 2000. As the gain is ***increased*** the system time response is ***faster*** and ***less damped***. The Bode diagram correspondingly shows ***larger bandwidths*** and ***larger resonant magnitudes***. Clearly, it is ***not possible*** to achieve low steady-state error and good transient response using only proportional control. As the gain is increased, the response becomes faster, but it has a lower phase margin. To remove the steady-state error and have better response, integral and/or derivative terms must be included in the controller.

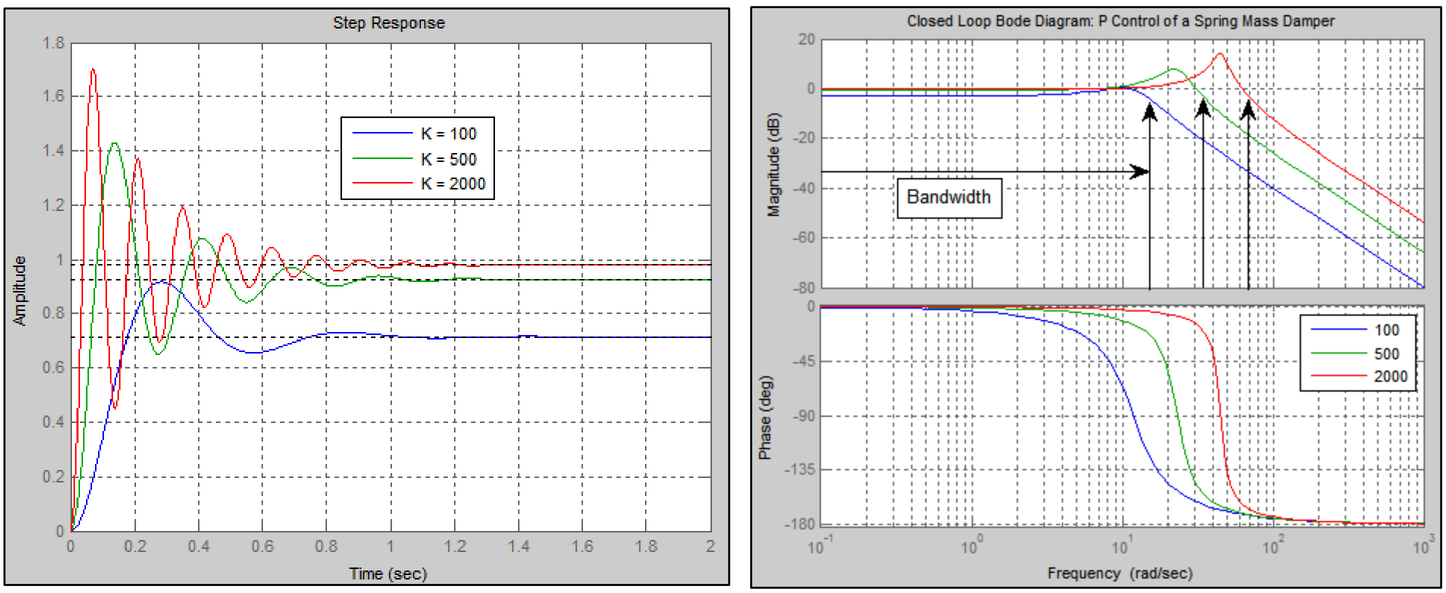


Figure 3. Closed Loop Step Response and Bode Diagrams for P Control

Proportional-Integral (PI) Control

If ***proportional-integral (PI) control*** is used, the controller transfer function is

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p(s+a)}{s} \qquad (4)$$

Here K_P and K_I represent the ***proportional*** and ***integral*** gains, and $a = K_I/K_P$ is the ratio of the integral and proportional gains. The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)} \qquad \frac{X}{X_d}(s) = \frac{K_P(s+a)}{s(s^2+8.8s+40)+K_P(s+a)} \qquad (5)$$

Integral control makes the system a ***type-one*** system, so the ***steady-state error*** due to a step input is ***zero***. This can be verified by using the final value theorem to show that $x_{ss} = 1$ when the input is a unit step function.

The root locus diagram for the closed-loop system (with $a = 3$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 4***. The root locus diagram also shows the locations of the closed-loop poles for a proportional gain $K_P = 50$. Note that the integral controller has ***added a third, slower pole*** to the system and has ***moved the asymptotes*** of the complex poles closer to the imaginary axis. For low gains, the system is ***slow and stable*** (first order dominant). As the gain is ***increased***, the system becomes ***faster*** with a ***decreasing phase margin***. The Bode diagram shows that the gain could be increased somewhat above $K_p = 25$ without significantly decreasing the stability margin. However, further increases will decrease the phase margin.

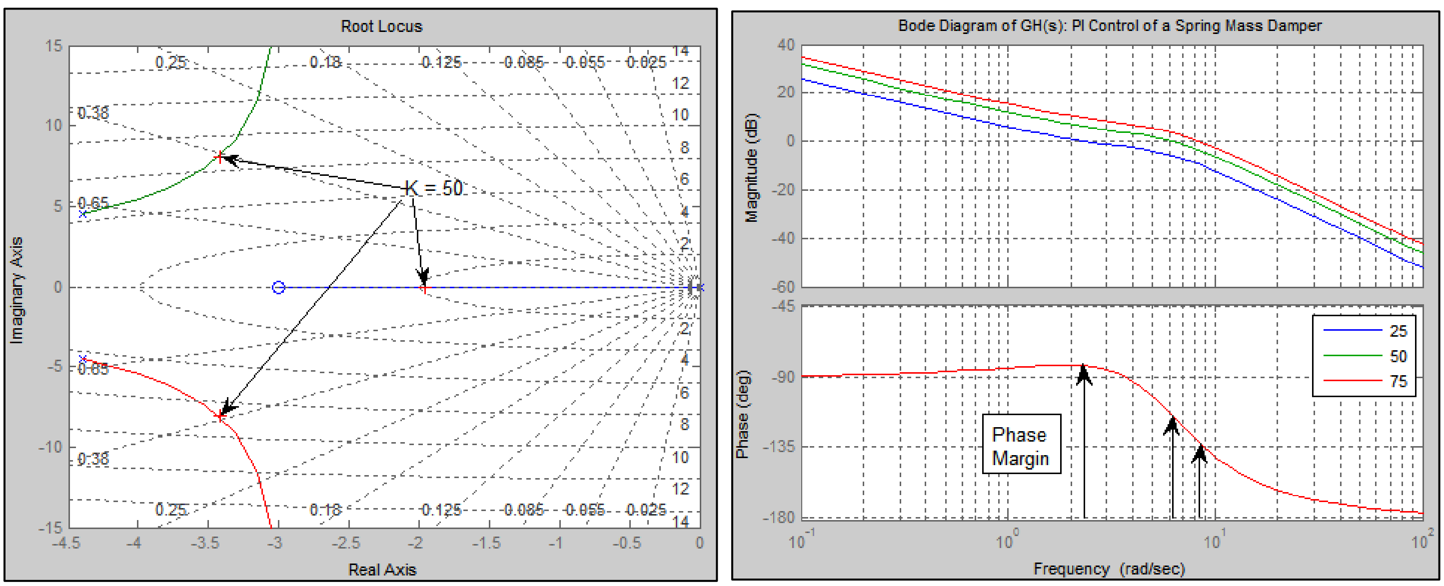


Figure 4. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PI Control ($a = 3$)

Fig. 5 shows step responses and Bode diagrams of the closed loop system for $a = 3$ and proportional gains of $K_p = 25$, 50, and 75. Integral control has ***removed*** the ***steady-state error*** and ***improved the transient response***, but it has also ***increased the system settling time***. Settling times can be lowered by increasing the gain. This will ***increase the system bandwidth***, but it will also decrease the stability margin.

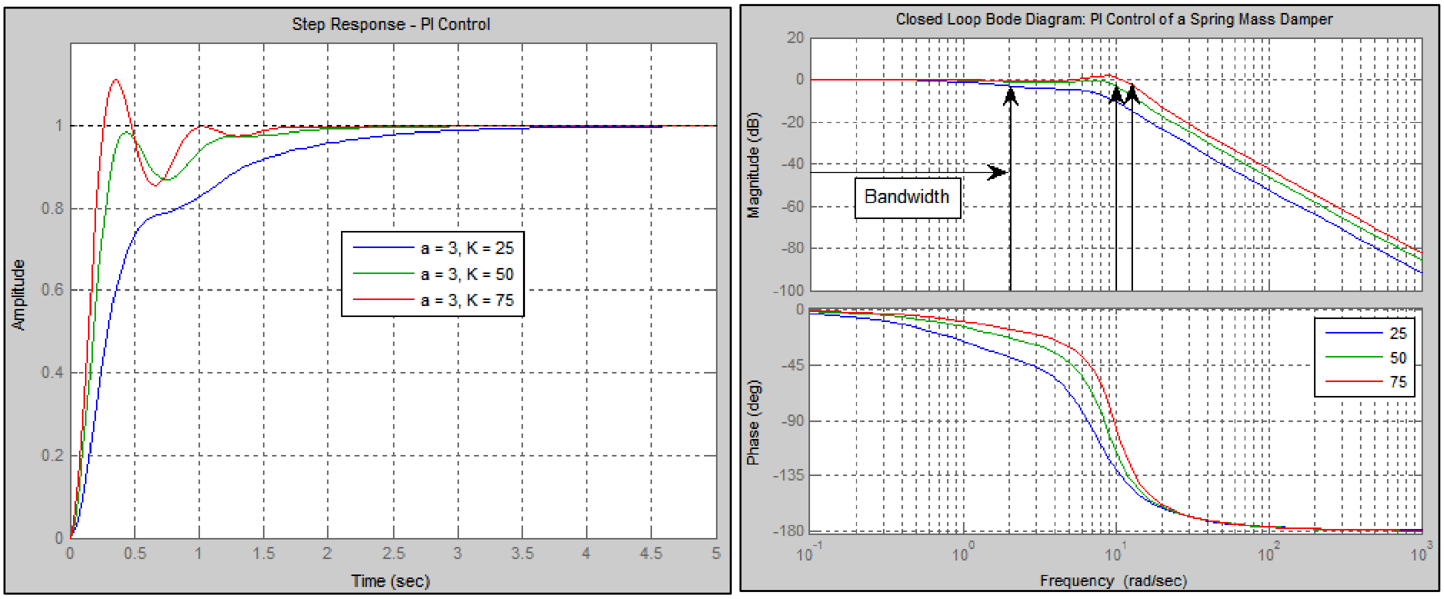


Figure 5. Closed Loop Step Response and Bode Diagrams for PI Control $(a=3)$

Proportional-Derivative (PD) Control

If ***proportional-derivative (PD) control*** is used, the controller transfer function is

$$\boxed{G_c(s)=K_p+K_D s=K_D(s+a)} \tag{6}$$

Here K_P and K_D represent the proportional and derivative gains, and $a=K_P/K_D$ is the ratio of the proportional and derivative gains. The loop and closed-loop transfer functions for this system are

$$\boxed{GH(s)=\frac{K_D(s+a)}{s^2+8.8s+40}} \qquad \boxed{\frac{X}{X_d}(s)=\frac{K_D(s+a)}{(s^2+8.8s+40)+K_D(s+a)}} \tag{7}$$

Without the integral control, this is again a ***type-zero*** system, and hence will have a ***finite steady-state error*** to a step input. Using the ***final-value theorem*** and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$x_{ss}=\lim_{s\to 0}\left(\$\cdot\frac{1}{\$}\cdot\frac{K_D(s+a)}{(s^2+8.8s+40)+K_D(s+a)}\right)=\frac{K_D a}{40+K_D a}=\frac{K_P}{40+K_P}<1 \tag{8}$$

As with simple proportional control, the ***larger the proportional gain***, the ***smaller the steady-state error***.

The root locus diagram for the closed-loop system (with $a=10$) for $K\geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 6***. The root locus diagram also shows the locations of the closed-loop poles for a derivative gain $K_D \approx 25.6$. As the gain is ***increased*** the system poles become ***faster*** and ***more damped***. The Bode diagram indicates that the phase margin never drops below 90 degrees indicating a very stable system for any gain.

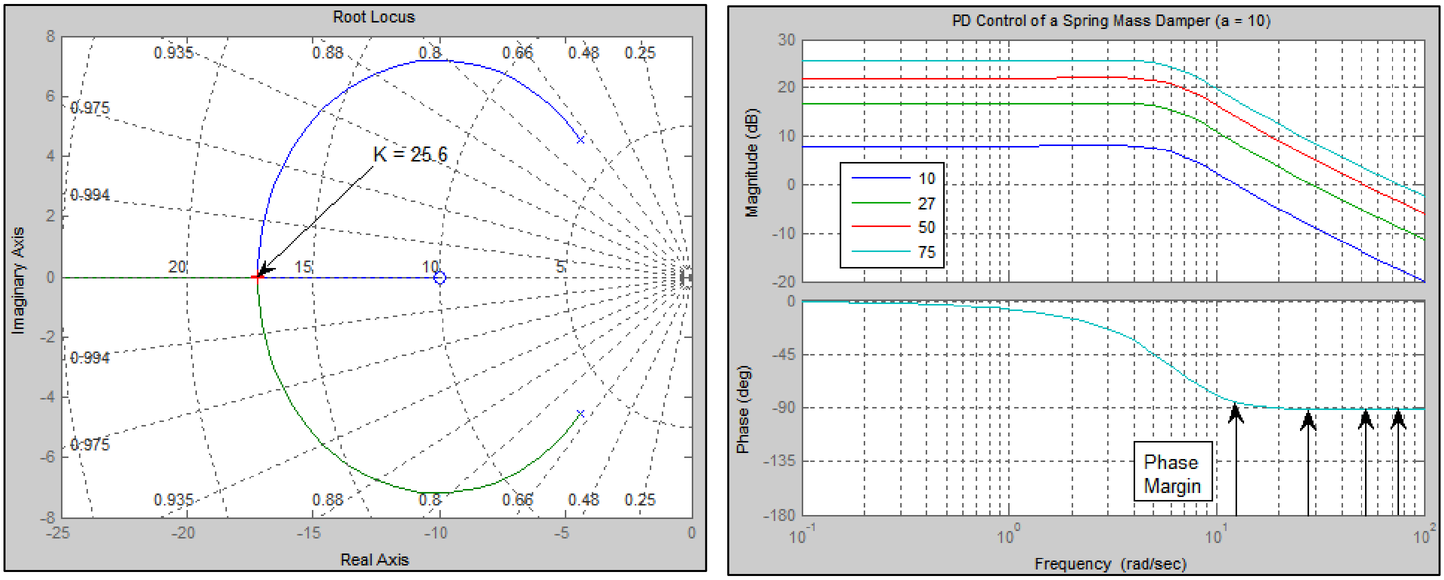


Figure 6. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PD Control $(a = 10)$

Fig. 7 shows step responses and Bode diagrams of the closed loop system for $a = 10$ and derivative gains of $K_D = 10$, 27, 50, and 75. The PD controller has ***decreased the system settling time*** considerably; however, to control the steady-state error, the derivative gain K_D must be high. This will ***decrease the response times and increase the bandwidth*** of the system and may make it ***susceptible to noise***.

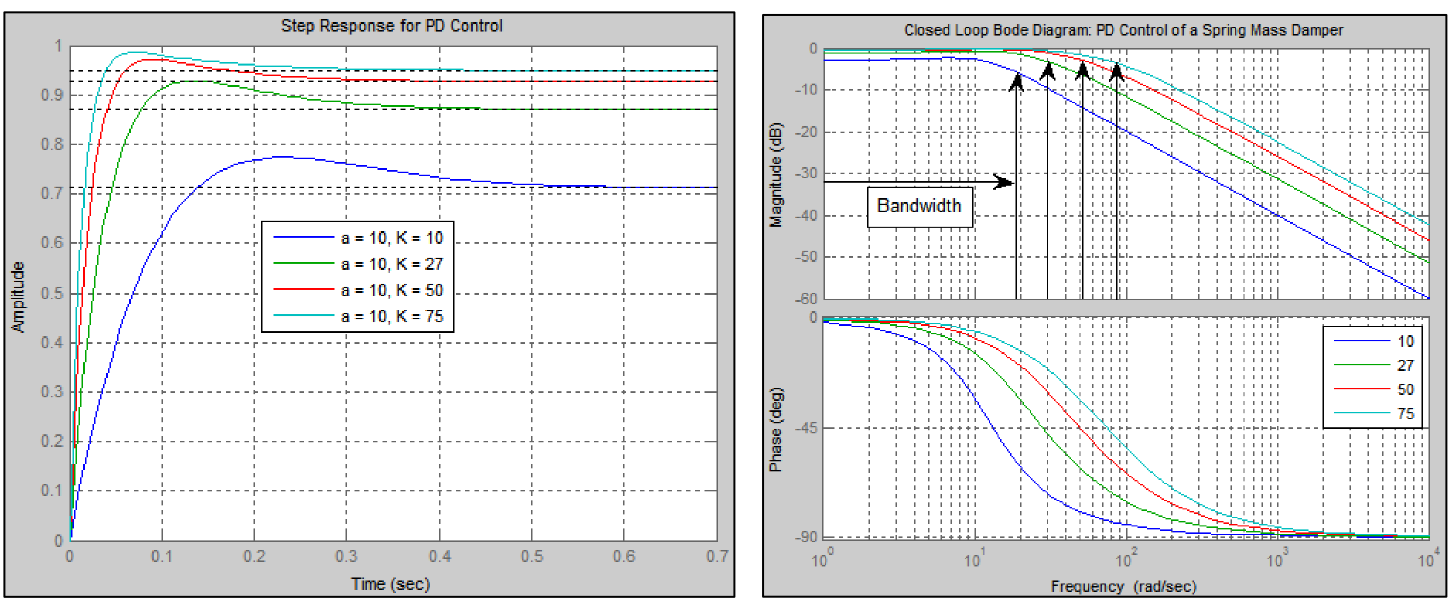


Figure 7. Closed Loop Step Response and Bode Diagrams for PD Control $(a = 10)$

Proportional-Integral-Derivative Control

If ***proportional-integral-derivative (PID) control*** is used, the controller transfer function is

$$G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s} \quad (9)$$

Here K_P, K_I, and K_D represent the proportional, integral, and derivative gains, $a = K_P / K_D$ is the ratio of the proportional and derivative gains, and $b = K_I / K_D$ is the ratio of the integral and derivative gains. The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)} \qquad \frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)} \quad (10)$$

Again, with integral control, the system is ***type-one*** and has zero steady-state error for a step input.

The root locus diagram for the closed-loop system (with $a = 15$ and $b = 50$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 8***. The locations of the closed-loop poles for $K_D \approx 15.8$ are also shown. As the gain is ***increased***, the system becomes ***faster without significant losses in the phase margin***.

Fig. 9 shows step responses and Bode diagrams of the closed-loop system for $a = 15$, $b = 50$, and gains $K_D = 5$, 10, and 15. Using both integral and derivative control has ***removed steady-state error*** and ***decreased system settling times*** while maintaining a reasonable transient response.

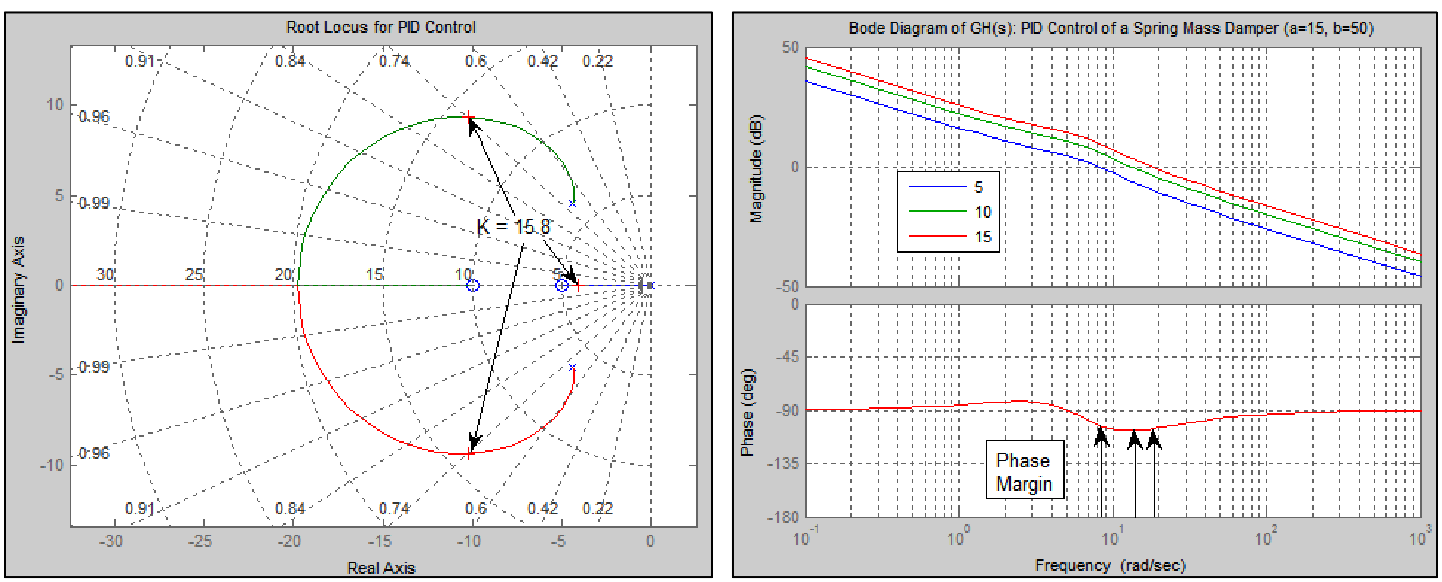


Figure 8. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PID Control ($a = 15, b = 50$)

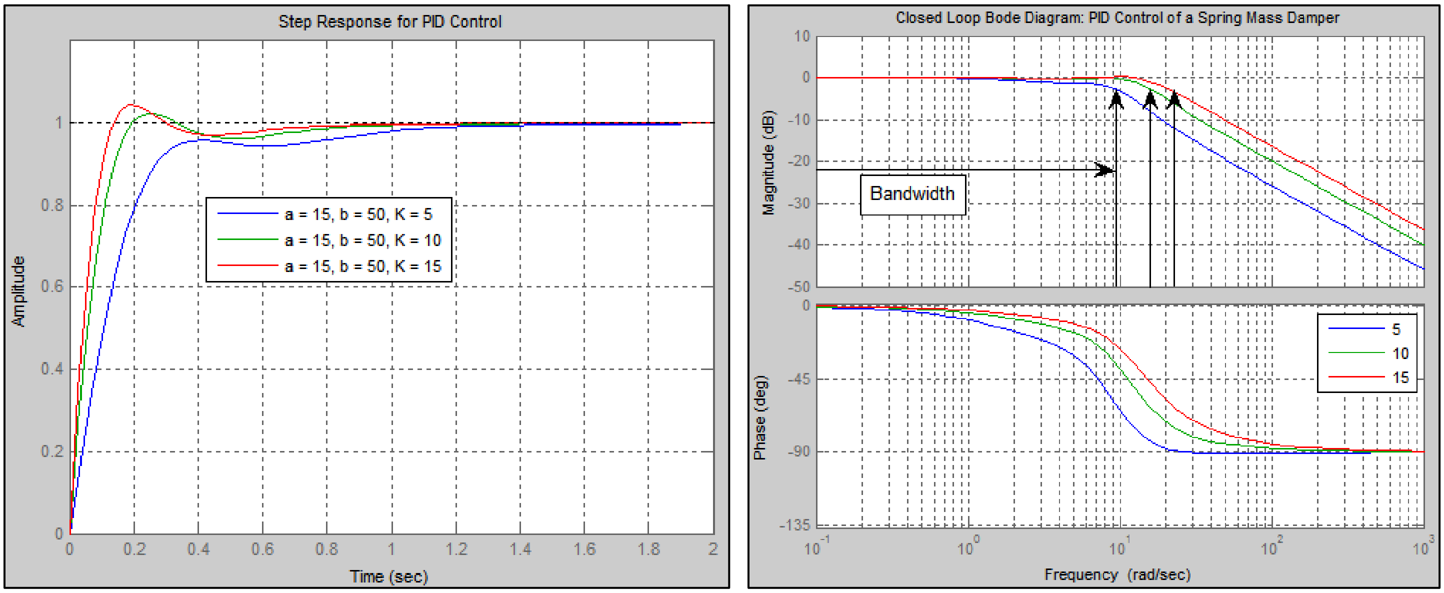


Figure 9. Closed-Loop Step Response and Bode Diagrams for PID Control $(a = 15, b = 50)$