Multibody Dynamics
Angular Velocity & Partial Angular Velocity Using Absolute Coordinates

Angular Velocity: 1-2-3 Rotation Sequence $(\theta_1, \theta_2, \theta_3)$

Given a rigid body $B:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$ moving in a fixed reference frame $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$, a set of intermediate reference frames $R':\left(\underset{\sim}{N}'_1, \underset{\sim}{N}'_2, \underset{\sim}{N}'_3\right)$ and $R'':\left(\underset{\sim}{N}''_1, \underset{\sim}{N}''_2, \underset{\sim}{N}''_3\right)$ can be defined as in previous notes. Using a 1-2-3 orientation angle sequence, the angular velocity of the body may then be written as

$$ {}^R\underset{\sim}{\omega}_B = {}^R\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_B = \dot{\theta}_1 \underset{\sim}{N}_1 + \dot{\theta}_2 \underset{\sim}{N}'_2 + \dot{\theta}_3 \underset{\sim}{N}''_3 $$

Using the definitions of the transformation matrices presented earlier, write

$$ \boxed{\left\{\underset{\sim}{N}'\right\} = [R_1]\left\{\underset{\sim}{N}\right\}} \quad \text{and} \quad \boxed{\left\{\underset{\sim}{N}''\right\} = [R_2]\left\{\underset{\sim}{N}'\right\} = [R_2][R_1]\left\{\underset{\sim}{N}\right\}} $$

The ***fixed-frame components*** of $\underset{\sim}{N}'_2$ are given by the second row of $[R_1]$, and the ***fixed-frame components*** $\underset{\sim}{N}''_3$ are given by the third row of $[R_2][R_1]$. Hence, the fixed-frame components of the angular velocity vector in matrix-vector form as

$$ \boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_2 \\ 0 & C_1 & -S_1 C_2 \\ 0 & S_1 & C_1 C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}} \qquad \text{"Fixed-Frame Components"} \qquad (1) $$

Note that the ***first column*** of the coefficient matrix holds the fixed-frame components of $\underset{\sim}{N}_1$, the ***second column*** holds the fixed-frame components of $\underset{\sim}{N}'_2$, and the ***third column*** holds the fixed-frame components of $\underset{\sim}{N}''_3$.

The same approach can be used to determine an equation for the ***body-frame components***. Or, the results can be found directly in Appendix II of Kane, Likins, and Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983. Using those results,

$$ \boxed{\begin{Bmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{Bmatrix} = \begin{bmatrix} C_2 C_3 & S_3 & 0 \\ -C_2 S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}} \qquad \text{"Body-Frame Components"} \qquad (2) $$

Partial Angular Velocities Using Angle Derivatives as Generalized Speeds

Using the orientation angle derivatives as the generalized speeds, the ***partial angular velocities*** of B are the partial derivatives of ${}^R\underset{\sim}{\omega}_B$ with respect to $\dot{\theta}_i$ $(i=1,2,3)$.

$$\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_1} = \underset{\sim}{N}_1 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_2} = \underset{\sim}{N}'_2 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_3} = \underset{\sim}{N}''_3$$

These results can be conveniently expressed in ***fixed-frame*** or ***body-frame*** components. Using ***fixed-frame*** components,

$$\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_1} \to \left\{\omega_{B,\dot{\theta}_1}\right\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_2} \to \left\{\omega_{B,\dot{\theta}_2}\right\} = \begin{Bmatrix} 0 \\ C_1 \\ S_1 \end{Bmatrix} \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}_3} \to \left\{\omega_{B,\dot{\theta}_3}\right\} = \begin{Bmatrix} S_2 \\ -S_1C_2 \\ C_1C_2 \end{Bmatrix}$$

These results can be expressed in a single matrix equation as

$$\left[\omega_{B,\dot{\theta}}\right] = \begin{bmatrix} 1 & 0 & S_2 \\ 0 & C_1 & -S_1C_2 \\ 0 & S_1 & C_1C_2 \end{bmatrix} \qquad \text{"Fixed-Frame Components"} \qquad (3)$$

Here, $\left[\omega_{B,\dot{\theta}}\right]$ is the ***partial angular velocity matrix*** of body B expressed using fixed-frame components.

Using the same process, the body-frame components of the partial velocity vectors can be written as a single matrix as

$$\left[\omega'_{B,\dot{\theta}}\right] = \begin{bmatrix} C_2C_3 & S_3 & 0 \\ -C_2S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix} \qquad \text{"Body-Frame Components"} \qquad (4)$$

As in previous notes, a "prime" has been used to indicate body-frame components.

***<u>Note</u>*:**

The specific entries of the partial angular velocity matrices depend on the choice of components (e.g. fixed-frame, body-frame, etc.), and they also depend on the chosen orientation angle sequence.

Partial Angular Velocities Using Angular Velocity Components as Generalized Speeds

Consider now using the angular velocity components as the generalized speeds for a body.

Fixed-Frame Components

Using the fixed-frame components of ${}^R\underset{\sim}{\omega}_B$ as the generalized speeds, the partial angular velocities are

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_1} = \underset{\sim}{N}_1} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_2} = \underset{\sim}{N}_2} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega_3} = \underset{\sim}{N}_3}$$

and the partial angular velocity matrix is

$$\boxed{\left[\omega_{B,\omega}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \text{"Fixed-Frame Components"} \qquad (5)$$

Body-Frame Components

Using body-frame components of ${}^R\underset{\sim}{\omega}_B$ as the generalized speeds, the partial angular velocities are

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_1} = \underset{\sim}{e}_1} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_2} = \underset{\sim}{e}_2} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \omega'_3} = \underset{\sim}{e}_3}$$

and the partial angular velocity matrix is

$$\boxed{\left[\omega_{B,\omega'}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \text{"Body-Frame Components"} \qquad (6)$$

Notes:

1. It is obvious from these results that this choice of generalized speeds simplifies the partial angular velocity matrices.
2. The entries of these partial velocity matrices are not dependent on what method is used to describe the orientation of the body.

Angular Velocity and Partial Angular Velocity Using Euler Parameters

If Euler parameters are used to describe the angular motion of a rigid body B, then from previous notes, the ***fixed-frame*** and ***body-frame components*** of the angular velocity can be written as

$$\boxed{\{\omega\} = 2[E]\{\dot{\varepsilon}\}} \qquad \text{"Fixed-Frame Components"}$$

$$\boxed{\{\omega'\} = 2[E']\{\dot{\varepsilon}\}} \qquad \text{"Body-Frame Components"}$$

Recall that in this case, $\{\omega\}$ and $\{\omega'\}$ are 4×1 vectors. The first three elements of the vectors are the components of ${}^{R}\underset{\sim}{\omega}_{B}$ and the last element is ***zero***.

From these equations, we could define a set of partial angular velocities associated with the Euler parameters. These are not particularly useful, however, because the Euler parameters are not ***independent***. In this case, it is natural to use the angular velocity components as the generalized speeds as discussed above. This, in turn, leads to the use of Kane's equations to formulate the equations of motion.