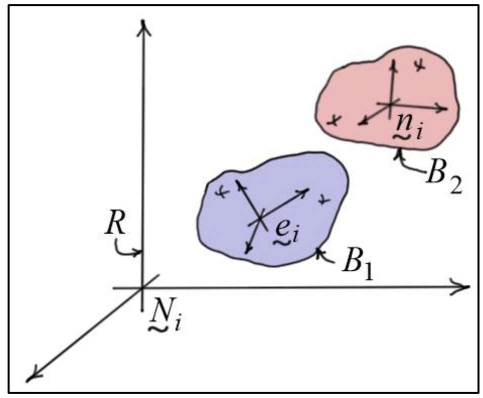
Multibody Dynamics
Coordinate Transformation Matrices Using Relative Coordinates

Instead of describing the orientation of a body relative to the fixed-frame frame, it may be more convenient to describe its orientation ***relative to another body*** in the system. In such a situation, coordinate transformation matrices can be defined from each body to the inertial frame and from one body to another.



Consider the ***two-body system*** shown with three reference frames: $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, $B_1:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$, and $B_2:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$. Using these frames, define ***four*** coordinate ***transformation matrices***, $\left[R_{B_1}\right]$, $\left[R_{B_2}\right]$, $\left[{}^{B_1}R_{B_2}\right]$, and $\left[{}^{B_2}R_{B_1}\right]$ by the following equations.

$$\boxed{\left\{\underset{\sim}{e}\right\}=\left[R_{B_1}\right]\left\{\underset{\sim}{N}\right\}}\quad \boxed{\left\{\underset{\sim}{n}\right\}=\left[R_{B_2}\right]\left\{\underset{\sim}{N}\right\}}\quad \boxed{\left\{\underset{\sim}{n}\right\}=\left[{}^{B_1}R_{B_2}\right]\left\{\underset{\sim}{e}\right\}}\quad \boxed{\left\{\underset{\sim}{e}\right\}=\left[{}^{B_2}R_{B_1}\right]\left\{\underset{\sim}{n}\right\}}$$

From the last two equations, and recalling these matrices are orthogonal, it is clear that $\left[{}^{B_1}R_{B_2}\right]^T=\left[{}^{B_2}R_{B_1}\right]$. Also, using the first three equations, it can be written

$$\left\{\underset{\sim}{n}\right\}=\left[R_{B_2}\right]\left\{\underset{\sim}{N}\right\}=\left[{}^{B_1}R_{B_2}\right]\left\{\underset{\sim}{e}\right\}=\left[{}^{B_1}R_{B_2}\right]\left[R_{B_1}\right]\left\{\underset{\sim}{N}\right\}$$

So,

$$\boxed{\left[R_{B_2}\right]=\left[{}^{B_1}R_{B_2}\right]\left[R_{B_1}\right]}$$

This result is ***easily extended*** to include as ***many bodies*** as necessary to move from a body frame to a fixed frame ***through*** frames of a ***series of interconnected bodies*** of the system.

$$\boxed{\left[R_{B_i}\right]=\left[{}^{B_{i-1}}R_{B_i}\right]\left[{}^{B_{i-2}}R_{B_{i-1}}\right]\cdots\left[{}^{B_1}R_{B_2}\right]\left[R_{B_1}\right]}$$

Notes:

1. The transformation matrix $\left[R_{B_i}\right]$ converts fixed-frame components into components in frame B_i.
2. The transformation matrix $\left[{}^{B_j}R_{B_i}\right]$ converts components in frame B_j into components in frame B_i.