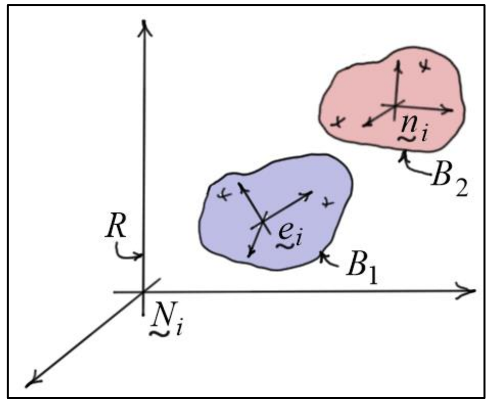
Multibody Dynamics
Angular Velocity and Partial Angular Velocity Using Relative Coordinates

Angular Velocity: 1-2-3 Rotation Sequence



Consider again the ***two-body system*** shown. It may be convenient at times to express the ***angular motion*** of body B_2 using the ***summation rule*** for angular velocities. For example, write

$$ {}^{R}\underset{\sim}{\omega}_{B_2} = {}^{R}\underset{\sim}{\omega}_{B_1} + {}^{B_1}\underset{\sim}{\omega}_{B_2} $$

Using a 1-2-3 angle rotation sequence and following earlier notes, the fixed-frame components of ${}^{R}\underset{\sim}{\omega}_{B_2}$ can be written as

$$ \left\{\omega_{B_2}\right\} = \left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\} \qquad \text{"Fixed-Frame Components"} \qquad (1) $$

where $\left\{\omega_{B_1}\right\}$ and $\left\{\omega_{B_2}\right\}$ represent the fixed-frame components of the angular velocities of bodies B_1 and B_2 in R, and $\left\{\hat{\omega}_{B_2}\right\}$ represents the B_1 components of the angular velocity of B_2 ***relative*** to B_1. Specifically,

$$ \left\{\omega_{B_1}\right\} = \begin{bmatrix} 1 & 0 & S_{12} \\ 0 & C_{11} & -S_{11}C_{12} \\ 0 & S_{11} & C_{11}C_{12} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{11} \\ \dot{\theta}_{12} \\ \dot{\theta}_{13} \end{Bmatrix} \quad \text{and} \quad \left\{\hat{\omega}_{B_2}\right\} = \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{21} \\ \dot{\theta}_{22} \\ \dot{\theta}_{23} \end{Bmatrix} \qquad (2) $$

Here, the first subscript is the "***body indicator***" and the second subscript is the "***variable indicator***". Also, the orientation angles $\theta_{1i}\ (i=1,2,3)$ of B_1 are measured ***relative to the fixed frame*** R, and the orientation angles $\theta_{2i}\ (i=1,2,3)$ of B_2 are measured ***relative to the moving frame*** B_1.

Using this approach, the absolute angular velocity components $\left\{\omega_{B_1}\right\}$ of B_1 are expressed in the fixed frame, and the relative angular velocity components $\left\{\hat{\omega}_{B_2}\right\}$ are expressed in the frame of the adjacent body B_1. In each case, the angular velocity components are expressed in the same frame in which the body orientation angles are measured.

Alternatively, the angular velocity components could be expressed in the body frames. For example, the B_2 components of ${}^{R}\underset{\sim}{\omega}_{B_2}$ can be written as

$$\left\{\omega'_{B_2}\right\} = \left[{}^{B_1}R_{B_2}\right]\left\{\omega'_{B_1}\right\} + \left\{\hat{\omega}'_{B_2}\right\} \qquad \text{“}B_2\text{ Components”} \tag{3}$$

Here,

$$\left\{\omega'_{B_1}\right\} = \begin{bmatrix} C_{12}C_{13} & S_{13} & 0 \\ -C_{12}S_{13} & C_{13} & 0 \\ S_{12} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_{11} \\ \dot{\theta}_{12} \\ \dot{\theta}_{13} \end{Bmatrix} \quad \text{and} \quad \left\{\hat{\omega}'_{B_2}\right\} = \begin{bmatrix} C_{22}C_{23} & S_{23} & 0 \\ -C_{22}S_{23} & C_{23} & 0 \\ S_{22} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_{21} \\ \dot{\theta}_{22} \\ \dot{\theta}_{23} \end{Bmatrix} \tag{4}$$

Using this approach, the absolute angular velocity components $\left\{\omega'_{B_1}\right\}$ of B_1 are expressed in the B_1 frame, and the relative angular velocity components $\left\{\hat{\omega}'_{B_2}\right\}$ are expressed in the B_2 frame. In each of the cases, the angular velocity components are expressed in the body frames.

Notes:

1. Motions between adjoining bodies of a system are more naturally described in terms of relative coordinates.
2. Unfortunately, the equations associated with the kinematics of the system are usually more complex when written in terms of relative coordinates.

Partial Angular Velocities Using Angle Derivatives as Generalized Speeds

Using Eqs. (1) and (2), the ***fixed-frame components*** of the ***partial angular velocity matrices*** for each of the two bodies can be written as

$$\left[\omega_{B_1,\dot{\theta}_{B_1}}\right] = \begin{bmatrix} 1 & 0 & S_{12} \\ 0 & C_{11} & -S_{11}C_{12} \\ 0 & S_{11} & C_{11}C_{12} \end{bmatrix} \qquad \left[\omega_{B_1,\dot{\theta}_{B_2}}\right] = \left[\ 0\ \right]_{3\times 3}$$

$$\left[\omega_{B_2,\dot{\theta}_{B_1}}\right] = \left[\omega_{B_1,\dot{\theta}_{B_1}}\right] \qquad \left[\omega_{B_2,\dot{\theta}_{B_2}}\right] = \left[R_{B_1}\right]^T \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix}$$

Using Eqs. (3) and (4), the ***body-frame components*** of the partial angular velocity matrices for each of the two bodies can be written as

B_1 components:

$$\left[\omega'_{B_1,\dot{\theta}_{B_1}}\right]=\begin{bmatrix} C_{12}C_{13} & S_{13} & 0 \\ -C_{12}S_{13} & C_{13} & 0 \\ S_{12} & 0 & 1 \end{bmatrix} \qquad \left[\omega'_{B_1,\dot{\theta}_{B_2}}\right]=\left[\ 0\ \right]_{3\times 3}$$

B_2 components:

$$\left[\omega'_{B_2,\dot{\theta}_{B_1}}\right]=\left[{}^{B_1}R_{B_2}\right]\begin{bmatrix} C_{12}C_{13} & S_{13} & 0 \\ -C_{12}S_{13} & C_{13} & 0 \\ S_{12} & 0 & 1 \end{bmatrix} \qquad \left[\omega'_{B_2,\dot{\theta}_{B_2}}\right]=\begin{bmatrix} C_{22}C_{23} & S_{23} & 0 \\ -C_{22}S_{23} & C_{23} & 0 \\ S_{22} & 0 & 1 \end{bmatrix}$$

<u>**Partial Angular Velocities Using Angular Velocity Components as Generalized Speeds**</u>

Using Eq. (1), the ***fixed-frame components*** of the ***partial angular velocity matrices*** for each of the two bodies can be written as

$$\left[\omega_{B_1,\omega_{B_1}}\right]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\omega_{B_1,\hat{\omega}_{B_2}}\right]=\left[\ 0\ \right]_{3\times 3}$$

$$\left[\omega_{B_2,\omega_{B_1}}\right]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\omega_{B_2,\hat{\omega}_{B_2}}\right]=\left[R_{B_1}\right]^T$$

Using Eq. (3), the ***body-frame components*** of the ***partial angular velocity matrices*** for each of the bodies can be written as

B_1 Components:

$$\left[\omega'_{B_1,\omega'_{B_1}}\right]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\omega'_{B_1,\hat{\omega}'_{B_2}}\right]=\left[\ 0\ \right]_{3\times 3}$$

B_2 Components:

$$\left[\omega'_{B_2,\omega'_{B_1}}\right]=\left[{}^{B_1}R_{B_2}\right] \qquad \left[\omega'_{B_2,\hat{\omega}'_{B_2}}\right]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$