Multibody Dynamics
Angular Acceleration Using Absolute Coordinates

Angle Derivatives as Generalized Speeds: 1-2-3 Rotation Sequence

Recall that the fixed-frame and body-frame components of ${}^R\underset{\sim}{\omega}_B$ the angular velocity of a body can be written in matrix-vector form as

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_2 \\ 0 & C_1 & -S_1 C_2 \\ 0 & S_1 & C_1 C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \left[\omega_{B,\dot{\theta}} \right] \left\{ \dot{\theta} \right\} \qquad \text{"Fixed-frame Components"} \qquad (1)$$

$$\begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{Bmatrix} = \begin{bmatrix} C_2 C_3 & S_3 & 0 \\ -C_2 S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \left[\omega_{B,\dot{\theta}}' \right] \left\{ \dot{\theta} \right\} \qquad \text{"Body-frame Components"} \qquad (2)$$

Consequently, the fixed-frame and body-frame components of ${}^R\underset{\sim}{\alpha}_B$ the angular acceleration of the body can be written in the following matrix-vector forms. Recall that the angular velocity vector can be differentiated in either the fixed-frame or the body-frame to find the angular acceleration.

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} = \left[\omega_{B,\dot{\theta}} \right] \left\{ \ddot{\theta} \right\} + \left[\dot{\omega}_{B,\dot{\theta}} \right] \left\{ \dot{\theta} \right\} \qquad \text{"Fixed-frame Components"} \qquad (3)$$

$$\begin{Bmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{Bmatrix} = \begin{Bmatrix} \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} = \left[\omega_{B,\dot{\theta}}' \right] \left\{ \ddot{\theta} \right\} + \left[\dot{\omega}_{B,\dot{\theta}}' \right] \left\{ \dot{\theta} \right\} \qquad \text{"Body-frame Components"} \qquad (4)$$

The time derivatives of the partial angular velocity matrices are

$$\left[\dot{\omega}_{B,\dot{\theta}} \right] = \begin{bmatrix} 0 & 0 & \dot{\theta}_2 C_2 \\ 0 & -\dot{\theta}_1 S_1 & \left(S_1 S_2 \dot{\theta}_2 - C_1 C_2 \dot{\theta}_1 \right) \\ 0 & \dot{\theta}_1 C_1 & -\left(S_1 C_2 \dot{\theta}_1 + C_1 S_2 \dot{\theta}_2 \right) \end{bmatrix} \qquad \left[\dot{\omega}_{B,\dot{\theta}}' \right] = \begin{bmatrix} -\left(S_2 C_3 \dot{\theta}_2 + C_2 S_3 \dot{\theta}_3 \right) & C_3 \dot{\theta}_3 & 0 \\ \left(S_2 S_3 \dot{\theta}_2 - C_2 C_3 \dot{\theta}_3 \right) & -S_3 \dot{\theta}_3 & 0 \\ C_2 \dot{\theta}_2 & 0 & 0 \end{bmatrix}$$

Angular Velocity Components as Generalized Speeds

If the angular velocity components are used as generalized speeds, then we have a much simpler form of the angular acceleration. Recalling that the partial velocity matrices associated with the angular velocity components are 3×3 identity matrices,

$$\boxed{\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} = \left[\omega_{B,\omega}\right]\left\{\dot{\omega}\right\} + \left[\dot{\omega}_{B,\omega}\right]\left\{\omega\right\} = \left[\omega_{B,\omega}\right]\left\{\dot{\omega}\right\} = \left\{\dot{\omega}\right\}}$$

"Fixed-frame Components"

$$\boxed{\begin{Bmatrix}\alpha_1'\\ \alpha_2'\\ \alpha_3'\end{Bmatrix} = \left[\omega'_{B,\omega'}\right]\left\{\dot{\omega}'\right\} + \left[\dot{\omega}'_{B,\omega'}\right]\left\{\omega'\right\} = \left[\omega'_{B,\omega'}\right]\left\{\dot{\omega}'\right\} = \left\{\dot{\omega}'\right\}}$$

"Body-frame Components"