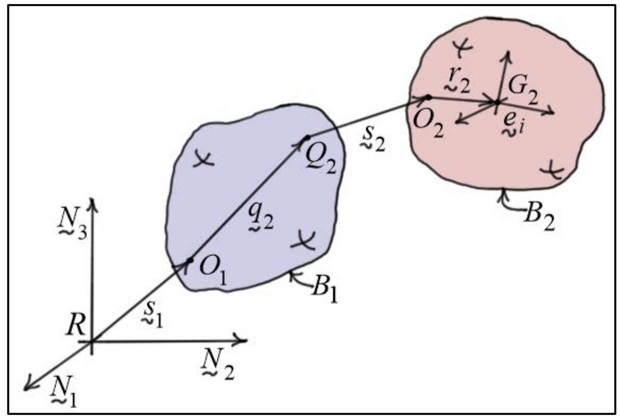
Multibody Dynamics
Velocities and Partial Velocities Using Relative Coordinates

Velocities

The motions of the mass centers of the bodies can also be determined by working ***through the other bodies*** of the system. For example, the fixed-frame components of the ***position vector*** of the mass center G_2 can be written as



$$\left\{p_{G_2}\right\}=\left\{s_1\right\}+\left\{q_2\right\}+\left\{s_2\right\}+\left\{r_2\right\}$$
$$=\left\{s_1\right\}+\left[R_{B_1}\right]^T\left\{q_2'\right\}+\left[R_{B_1}\right]^T\left\{s_2'\right\}+\left[R_{B_2}\right]^T\left\{r_2'\right\}$$

Here it has been assumed that $\{s_1\}$ is given in the fixed frame, $\{q_2'\}$ and $\{s_2'\}$ are given in the B_1 frame, and $\{r_2'\}$ is given in the B_2 frame.

The ***velocity*** of G_2 can be found by ***differentiating*** the above equation.

$$\left\{v_{G_2}\right\}=\left\{\dot{s}_1\right\}+\left[\dot{R}_{B_1}\right]^T\left\{q_2'\right\}+\left[\dot{R}_{B_1}\right]^T\left\{s_2'\right\}+\left[R_{B_1}\right]^T\left\{\dot{s}_2'\right\}+\left[\dot{R}_{B_2}\right]^T\left\{r_2'\right\}$$

To expand this equation further, first decide whether to use ***fixed-frame*** or ***body-frame components*** for the ***angular velocity vectors***. The two results are as follows.

$$\boxed{\left\{v_{G_2}\right\}=\left\{\dot{s}_1\right\}+\left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\left(\left\{q_2'\right\}+\left\{s_2'\right\}\right)+\left[R_{B_1}\right]^T\left\{\dot{s}_2'\right\}+\left[\tilde{\omega}_{B_2}\right]\left[R_{B_2}\right]^T\left\{r_2'\right\}} \quad (1)$$

$$\boxed{\left\{v_{G_2}\right\}=\left\{\dot{s}_1\right\}+\left[R_{B_1}\right]^T\left[\tilde{\omega}_{B_1}'\right]\left(\left\{q_2'\right\}+\left\{s_2'\right\}\right)+\left[R_{B_1}\right]^T\left\{\dot{s}_2'\right\}+\left[R_{B_2}\right]^T\left[\tilde{\omega}_{B_2}'\right]\left\{r_2'\right\}} \quad (2)$$

The velocities of ***other points*** of B_2 are found by simply replacing the coordinates $\{r_2'\}$ with the coordinates of that point.

To find the ***partial velocities*** of G_2, first ***rearrange*** Eqs. (1) and (2). As a first step in this process, note that ${}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r} = -\underset{\sim}{r} \times {}^R\underset{\sim}{\omega}_B$. This vector identity can be written in matrix form as $\left[\tilde{\omega}_B\right]\{r\}=-\left[\tilde{r}\right]\left\{\omega_B\right\}$. Using this result in Eqs. (1) and (2), write the following.

Using fixed-frame components for the angular velocity vectors:

$$\begin{aligned}
\{v_{G_2}\} &= \{\dot{s}_1\} + [\tilde{\omega}_{B_1}][R_{B_1}]^T(\{q'_2\}+\{s'_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [\tilde{\omega}_{B_2}][R_{B_2}]^T\{r'_2\} \\
&= \{\dot{s}_1\} + [\tilde{\omega}_{B_1}](\{q_2\}+\{s_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [\tilde{\omega}_{B_2}]\{r_2\} \\
&= \{\dot{s}_1\} - ([\tilde{q}_2]+[\tilde{s}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2]\{\omega_{B_2}\} \\
&= \{\dot{s}_1\} - ([\tilde{q}_2]+[\tilde{s}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2]\left(\{\omega_{B_1}\} + [R_{B_1}]^T\{\hat{\omega}_{B_2}\}\right) \\
&= \{\dot{s}_1\} - ([\tilde{q}_2]+[\tilde{s}_2]+[\tilde{r}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2][R_{B_1}]^T\{\hat{\omega}_{B_2}\}
\end{aligned} \tag{3}$$

Recall here that the entries of $\{\omega_{B_1}\}$ are expressed in the fixed frame, and the entries of $\{\hat{\omega}_{B_2}\}$ are expressed in the B_1 frame. In both cases, these are the frames in which the angular velocities are measured.

Using body-frame components for the angular velocity vectors:

$$\begin{aligned}
\{v_{G_2}\} &= \{\dot{s}_1\} + [R_{B_1}]^T[\tilde{\omega}'_{B_1}](\{q'_2\}+\{s'_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [R_{B_2}]^T[\tilde{\omega}'_{B_2}]\{r'_2\} \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2]+[\tilde{s}'_2])\{\omega'_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [R_{B_2}]^T[\tilde{r}'_2]\{\omega'_{B_2}\} \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2]+[\tilde{s}'_2])\{\omega'_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} \\
&\qquad - [R_{B_2}]^T[\tilde{r}'_2]\left([{}^{B_2}R_{B_1}]^T\{\omega'_{B_1}\} + \{\hat{\omega}'_{B_2}\}\right) \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2]+[\tilde{s}'_2])\{\omega'_{B_1}\} - [R_{B_2}]^T[\tilde{r}'_2][{}^{B_2}R_{B_1}]^T\{\omega'_{B_1}\} \\
&\qquad + [R_{B_1}]^T\{\dot{s}'_2\} - [R_{B_2}]^T[\tilde{r}'_2]\{\hat{\omega}'_{B_2}\}
\end{aligned} \tag{4}$$

<u>Partial Velocities</u>

Eqs. (3) and (4) can be used to find the ***partial velocities*** of the mass centers of the bodies.

<u>Case 1:</u>

If the ***generalized speeds*** are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\{\omega_{B_1}\}$, and $\{\hat{\omega}_{B_2}\}$, then

$$\boxed{\left\{ v_{G_2} \right\} = \left[v_{G_2,\dot{s}_1} \right]\left\{ \dot{s}_1 \right\} + \left[v_{G_2,\omega_{B_1}} \right]\left\{ \omega_{B_1} \right\} + \left[v_{G_2,\dot{s}'_2} \right]\left\{ \dot{s}'_2 \right\} + \left[v_{G_2,\hat{\omega}_{B_2}} \right]\left\{ \hat{\omega}_{B_2} \right\}}$$

where

$\left[v_{G_2,\dot{s}_1} \right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\omega_{B_1}} \right] = -\left(\left[\tilde{q}_2 \right] + \left[\tilde{s}_2 \right] + \left[\tilde{r}_2 \right] \right)$$

$$\left[v_{G_2,\dot{s}'_2} \right] = \left[R_{B_1} \right]^T$$

$$\left[v_{G_2,\hat{\omega}_{B_2}} \right] = -\left[\tilde{r}_2 \right]\left[R_{B_1} \right]^T$$

Case 2:

If the ***generalized speeds*** are defined to be the components of $\left\{ \dot{s}_1 \right\}$, $\left\{ \dot{s}'_2 \right\}$, $\left\{ \omega'_{B_1} \right\}$, and $\left\{ \hat{\omega}'_{B_2} \right\}$, then

$$\boxed{\left\{ v_{G_2} \right\} = \left[v_{G_2,\dot{s}_1} \right]\left\{ \dot{s}_1 \right\} + \left[v_{G_2,\omega'_{B_1}} \right]\left\{ \omega'_{B_1} \right\} + \left[v_{G_2,\dot{s}'_2} \right]\left\{ \dot{s}'_2 \right\} + \left[v_{G_2,\hat{\omega}'_{B_2}} \right]\left\{ \hat{\omega}'_{B_2} \right\}}$$

Here,

$\left[v_{G_2,\dot{s}_1} \right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\omega'_{B_1}} \right] = -\left[R_{B_1} \right]^T \left(\left[\tilde{q}'_2 \right] + \left[\tilde{s}'_2 \right] \right) - \left[R_{B_2} \right]^T \left[\tilde{r}'_2 \right]\left[{}^{B_2}R_{B_1} \right]^T$$

$$\left[v_{G_2,\dot{s}'_2} \right] = \left[R_{B_1} \right]^T$$

$$\left[v_{G_2,\hat{\omega}'_{B_2}} \right] = -\left[R_{B_2} \right]^T \left[\tilde{r}'_2 \right]$$

Case 3:

If, instead, the generalized speeds are defined to be the components of $\left\{ \dot{s}_1 \right\}$, $\left\{ \dot{s}_2 \right\}$, and a set of ***orientation angle derivatives***, then we can write $\left\{ \omega_{B_1} \right\}$ and $\left\{ \hat{\omega}_{B_2} \right\}$ (or $\left\{ \omega'_{B_1} \right\}$ and $\left\{ \hat{\omega}'_{B_2} \right\}$) in terms of the orientation angles and then identify the partial velocities. Returning to Eq. (3), for example, write

$$
\begin{aligned}
\{v_{G_2}\} &= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2][R_{B_1}]^T\{\hat{\omega}_{B_2}\} \\
&= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]\{\dot{\theta}_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} \\
&\qquad - [\tilde{r}_2][R_{B_1}]^T\left[\hat{\omega}_{B_2,\dot{\theta}_{B_2}}\right]\{\dot{\theta}_{B_2}\} \\
&= \left[v_{G_2,\dot{s}_1}\right]\{\dot{s}_1\} + \left[v_{G_2,\dot{\theta}_{B_1}}\right]\{\dot{\theta}_{B_1}\} + \left[v_{G_2,\dot{s}'_2}\right]\{\dot{s}'_2\} + \left[v_{G_2,\dot{\theta}_{B_2}}\right]\{\dot{\theta}_{B_2}\}
\end{aligned}
\tag{5}
$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\dot{\theta}_{B_1}}\right] = -\left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]$$

$$\left[v_{G_2,\dot{s}'_2}\right] = [R_{B_1}]^T$$

$$\left[v_{G_2,\dot{\theta}_{B_2}}\right] = -[\tilde{r}_2][R_{B_1}]^T\left[\hat{\omega}_{B_2,\dot{\theta}_{B_2}}\right]$$

The partial angular velocity matrices are defined in previous notes for a 1-2-3 rotation sequence.