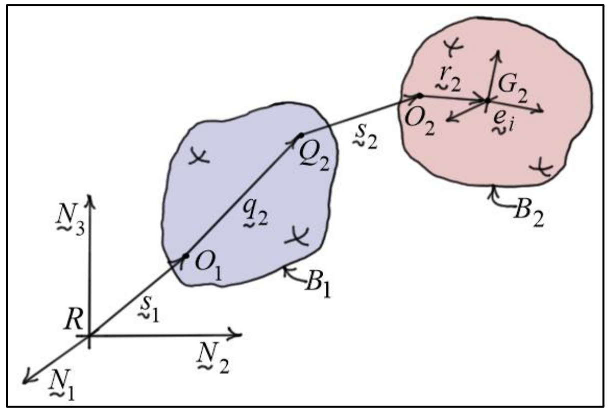
Multibody Dynamics
Accelerations Using Relative Coordinates

Velocities



Consider again the ***two-body system*** shown. In previous notes, it was found that the ***fixed-frame components*** of ${}^R\underset{\sim}{v}_{G_2}$ the ***velocity*** of G_2 in the fixed frame R can be written as

$$\left\{v_{G_2}\right\}=\left\{\dot{s}_1\right\}-\left(\left[\tilde{q}_2\right]+\left[\tilde{s}_2\right]+\left[\tilde{r}_2\right]\right)\left\{\omega_{B_1}\right\}+\left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\}-\left[\tilde{r}_2\right]\left[R_{B_1}\right]^T\left\{\hat{\omega}_{B_2}\right\} \tag{1}$$

or

$$\begin{aligned}\left\{v_{G_2}\right\}=&\left\{\dot{s}_1\right\}-\left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left\{\omega'_{B_1}\right\}-\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T\left\{\omega'_{B_1}\right\}\\&+\left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\}-\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left\{\hat{\omega}'_{B_2}\right\}\end{aligned} \tag{2}$$

The first equation is written in terms of the ***fixed-frame components*** of ${}^R\underset{\sim}{\omega}_{B_1}$ and the B_1 ***frame components*** of ${}^{B_1}\underset{\sim}{\omega}_{B_2}$, and the second is written in terms of the ***body-frame components*** of the angular velocities.

Accelerations

Case 1: Fixed-Frame Angular Velocity Components as Generalized Speeds

If the generalized speeds are defined to be the elements of $\left\{\dot{s}_1\right\}$, $\left\{\dot{s}'_2\right\}$, $\left\{\omega_{B_1}\right\}$, and $\left\{\hat{\omega}_{B_2}\right\}$, then write

$$\boxed{\left\{v_{G_2}\right\}=\left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\}+\left[v_{G_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\}+\left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\}+\left[v_{G_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}} \tag{3}$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$\left[v_{G_2,\omega_{B_1}}\right]=-\left(\left[\tilde{q}_2\right]+\left[\tilde{s}_2\right]+\left[\tilde{r}_2\right]\right)$

$\left[v_{G_2,\dot{s}'_2}\right]=\left[R_{B_1}\right]^T$

$\left[v_{G_2,\hat{\omega}_{B_2}}\right]=-\left[\tilde{r}_2\right]\left[R_{B_1}\right]^T$

Differentiating Eq. (3), the ***fixed-frame components*** of the ***acceleration*** of G_2 can then be written as

$$\begin{aligned}\left\{a_{G_2}\right\} &= \left[v_{G_2,\dot{s}_1}\right]\left\{\ddot{s}_1\right\} + \left[v_{G_2,\omega_{B_1}}\right]\left\{\dot{\omega}_{B_1}\right\} + \left[\dot{v}_{G_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\} \\ &\quad + \left[v_{G_2,\dot{s}'_2}\right]\left\{\ddot{s}'_2\right\} + \left[\dot{v}_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\hat{\omega}_{B_2}}\right]\left\{\dot{\hat{\omega}}_{B_2}\right\} + \left[\dot{v}_{G_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}\end{aligned} \tag{4}$$

where the time derivatives of the partial velocity matrices can be written as follows.

a) $\left[\dot{v}_{G_2,\omega_{B_1}}\right] = -\left(\left[\dot{\tilde{q}}_2\right] + \left[\dot{\tilde{s}}_2\right] + \left[\dot{\tilde{r}}_2\right]\right)$

The components of the vectors $\{\dot{q}_2\}$, $\{\dot{s}_2\}$, and $\{\dot{r}_2\}$ are elements of the matrices $\left[\dot{\tilde{q}}_2\right]$, $\left[\dot{\tilde{s}}_2\right]$, and $\left[\dot{\tilde{r}}_2\right]$. These components are found as follows.

$$\{\dot{q}_2\} = \frac{d}{dt}\left(\left[R_{B_1}\right]^T\{q'_2\}\right) = \left[\dot{R}_{B_1}\right]^T\{q'_2\} = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\{q'_2\} \tag{5}$$

$$\begin{aligned}\{\dot{s}_2\} &= \frac{d}{dt}\left(\left[R_{B_1}\right]^T\{s'_2\}\right) = \left[\dot{R}_{B_1}\right]^T\{s'_2\} + \left[R_{B_1}\right]^T\{\dot{s}'_2\} \\ &= \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\{s'_2\} + \left[R_{B_1}\right]^T\{\dot{s}'_2\}\end{aligned} \tag{6}$$

$$\{\dot{r}_2\} = \frac{d}{dt}\left(\left[R_{B_2}\right]^T\{r'_2\}\right) = \left[\dot{R}_{B_2}\right]^T\{r'_2\} = \left[\tilde{\omega}_{B_2}\right]\left[R_{B_2}\right]^T\{r'_2\} \tag{7}$$

b) $$\left[\dot{v}_{G_2,\dot{s}'_2}\right] = \left[\dot{R}_{B_1}\right]^T = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T \tag{8}$$

c) $$\left[\dot{v}_{G_2,\hat{\omega}_{B_2}}\right] = -\left[\dot{\tilde{r}}_2\right]\left[R_{B_1}\right]^T - \left[\tilde{r}_2\right]\left[\dot{R}_{B_1}\right]^T = -\left[\dot{\tilde{r}}_2\right]\left[R_{B_1}\right]^T - \left[\tilde{r}_2\right]\left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T \tag{9}$$

The elements of $\left[\dot{\tilde{r}}_2\right]$ are calculated in Eq. (7) above.

Case 2: Body-Frame Angular Velocity Components as Generalized Speeds

If the ***generalized speeds*** are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\left\{\omega'_{B_1}\right\}$, and $\left\{\hat{\omega}'_{B_2}\right\}$, then

$$\left\{v_{G_2}\right\} = \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\omega'_{B_1}}\right]\left\{\omega'_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\hat{\omega}'_{B_2}}\right]\left\{\hat{\omega}'_{B_2}\right\} \tag{10}$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\omega'_{B_1}}\right] = -\left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)-\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T$$

$$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2,\hat{\omega}'_{B_2}}\right] = -\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]$$

Differentiating Eq. (10), the ***fixed-frame components*** of the ***acceleration*** of G_2 can then be written as

$$\left\{a_{G_2}\right\} = \left[v_{G_2,\dot{s}_1}\right]\left\{\ddot{s}_1\right\}+\left[v_{G_2,\omega'_{B_1}}\right]\left\{\dot{\omega}'_{B_1}\right\}+\left[\dot{v}_{G_2,\omega'_{B_1}}\right]\left\{\omega'_{B_1}\right\}$$
$$+\left[v_{G_2,\dot{s}'_2}\right]\left\{\ddot{s}'_2\right\}+\left[\dot{v}_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\}+\left[v_{G_2,\hat{\omega}'_{B_2}}\right]\left\{\dot{\hat{\omega}}'_{B_2}\right\}+\left[\dot{v}_{G_2,\hat{\omega}'_{B_2}}\right]\left\{\hat{\omega}'_{B_2}\right\}$$

Here, the time derivatives of the partial velocity matrices can be written as follows.

a) $$\left[\dot{v}_{G_2,\omega'_{B_1}}\right] = -\left[R_{B_1}\right]^T\left[\dot{\tilde{s}}'_2\right]-\left[\dot{R}_{B_1}\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)-\left[\dot{R}_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T$$
$$-\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_1}\dot{R}_{B_2}\right]$$
$$= -\left[R_{B_1}\right]^T\left[\dot{\tilde{s}}'_2\right]-\left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)$$
$$-\left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T-\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[\tilde{\hat{\omega}}'_{B_2}\right]\left[{}^{B_1}R_{B_2}\right]$$

b) $$\left[\dot{v}_{G_2,\dot{s}'_2}\right] = \left[\dot{R}_{B_1}\right]^T = \left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]$$

c) $$\left[\dot{v}_{G_2,\hat{\omega}'_{B_2}}\right] = -\left[\dot{R}_{B_2}\right]^T\left[\tilde{r}'_2\right] = -\left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\left[\tilde{r}'_2\right]$$

Case 3: Orientation Angle Derivatives as Generalized Speeds

If, instead, the generalized speeds are defined to be the components of $\left\{\dot{s}_1\right\}$, $\left\{\dot{s}_2\right\}$, and a set of ***orientation angle derivatives***, then $\left\{\omega_{B_1}\right\}$ and $\left\{\hat{\omega}_{B_2}\right\}$ (or $\left\{\omega'_{B_1}\right\}$ and $\left\{\hat{\omega}'_{B_2}\right\}$) can be written in terms of the orientation angles. Returning to Eq. (1), for example,

$$
\begin{aligned}
\left\{v_{G_2}\right\} &= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left[R_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\} \\
&= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]\left\{\dot{\theta}_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left[\omega_{B_2,\dot{\theta}_{B_2}}\right]\left\{\dot{\theta}_{B_2}\right\} \\
&= \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\dot{\theta}_{B_1}}\right]\left\{\dot{\theta}_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\dot{\theta}_{B_2}}\right]\left\{\dot{\theta}_{B_2}\right\}
\end{aligned}
\qquad (11)
$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\dot{\theta}_{B_1}}\right] = -\left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]$$

$$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2,\dot{\theta}_{B_2}}\right] = -\left[\tilde{r}_2\right]\left[\omega_{B_2,\dot{\theta}_{B_2}}\right]$$

The partial angular velocity matrices are defined in previous notes for a 1-2-3 rotation sequence.

The fixed-frame components of the acceleration of G_2 can then be written as

$$
\begin{aligned}
\left\{a_{G_2}\right\} &= \left[v_{G_2,\dot{s}_1}\right]\left\{\ddot{s}_1\right\} + \left[v_{G_2,\dot{\theta}_{B_1}}\right]\left\{\ddot{\theta}_{B_1}\right\} + \left[\dot{v}_{G_2,\dot{\theta}_{B_1}}\right]\left\{\dot{\theta}_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\ddot{s}'_2\right\} + \left[\dot{v}_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} \\
&\quad + \left[v_{G_2,\dot{\theta}_{B_2}}\right]\left\{\ddot{\theta}_{B_2}\right\} + \left[\dot{v}_{G_2,\dot{\theta}_{B_2}}\right]\left\{\dot{\theta}_{B_2}\right\}
\end{aligned}
\qquad (12)
$$

Here,

$$\left[\dot{v}_{G_2,\dot{\theta}_{B_1}}\right] = -\left(\left[\dot{\tilde{q}}_2\right] + \left[\dot{\tilde{s}}_2\right] + \left[\dot{\tilde{r}}_2\right]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right] - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left[\dot{\omega}_{B_1,\dot{\theta}_{B_1}}\right]$$

$$\left[\dot{v}_{G_2,\dot{s}'_2}\right] = \left[\dot{R}_{B_1}\right]^T = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T$$

$$\left[\dot{v}_{G_2,\dot{\theta}_{B_2}}\right] = -\left[\dot{\tilde{r}}_2\right]\left[\omega_{B_2,\dot{\theta}_{B_2}}\right] - \left[\tilde{r}_2\right]\left[\dot{\omega}_{B_2,\dot{\theta}_{B_2}}\right]$$