

## Multibody Dynamics

### Connecting Joints – Part I

- Connecting joints **constrain the relative motion** between adjoining bodies in a multibody system. Joints range from allowing no relative motion (rigid joint) to allowing all motions (free joint).
- When degrees of freedom are removed, **constraint forces** and or **torques** result. When degrees of freedom are not removed, forces or torques at a joint may or may not be zero. For example, bodies may have interconnective springs and dampers that do not remove degrees of freedom but do restrict the motion by applying loads associated with the relative motions.
- A listing of **common connecting joints** is shown in the following table.

<i>Joint Type</i>	<i>Degrees of Freedom</i>
Rigid	0
Hinge (revolute)	1 (1 rotation)
Spherical (ball & socket)	3 (3 rotation)
Two-Angle (universal)	2 (2 rotation)
Prismatic (slider)	1 (1 translation)
Cylindrical	2 (1 translation, 1 rotation)
Free	6 (3 translation, 3 rotation)

- For every degree of freedom that a joint eliminates, a constraint equation can be written. The following paragraphs outline the form of the constraint equations for some common joints.

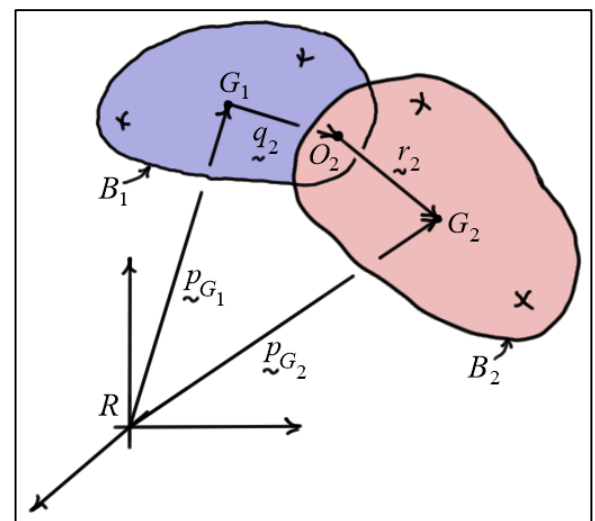
#### Spherical Joint: Absolute Coordinates

- A **spherical** (or ball-and-socket) joint allows two bodies to share a common point, but still rotate freely relative to each other. Using **absolute coordinates**, the constraint equation can be written as

$$\{p_{G_2}\} = \{p_{G_1}\} + [R_{B_1}]^T \{q'_2\} + [R_{B_2}]^T \{r'_2\}$$

or

$$\{p_{G_2}\} - \{p_{G_1}\} - [R_{B_1}]^T \{q'_2\} - [R_{B_2}]^T \{r'_2\} = \{0\}$$



(1)

- Eq. (1) represents a set of **three scalar constraint equations** that eliminate the three translational degrees of freedom between the bodies.
- For incorporation into a set of equations of motion, the constraint equations can be **differentiated** twice so they are in the form of second order differential equations. Using **fixed-frame** angular velocity components gives

$$\begin{aligned}
\{0\} &= \{v_{G_2}\} - \{v_{G_1}\} - [\dot{R}_{B_1}]^T \{q'_2\} - [\dot{R}_{B_2}]^T \{r'_2\} \\
&= \{v_{G_2}\} - \{v_{G_1}\} - [\tilde{\omega}_{B_1}] [R_{B_1}]^T \{q'_2\} - [\tilde{\omega}_{B_2}] [R_{B_2}]^T \{r'_2\} \\
&= \{v_{G_2}\} - \{v_{G_1}\} - [\tilde{\omega}_{B_1}] \{q_2\} - [\tilde{\omega}_{B_2}] \{r_2\} \\
&= \{v_{G_2}\} - \{v_{G_1}\} + [\tilde{q}_2] \{\omega_{B_1}\} + [\tilde{r}_2] \{\omega_{B_2}\}
\end{aligned}$$

- Differentiating again gives

$$\boxed{\{0\} = \{a_{G_2}\} - \{a_{G_1}\} + [\tilde{q}_2] \{\dot{\omega}_{B_1}\} + [\dot{\tilde{q}}_2] \{\omega_{B_1}\} + [\tilde{r}_2] \{\dot{\omega}_{B_2}\} + [\dot{\tilde{r}}_2] \{\omega_{B_2}\}} \quad (2)$$

Calculation of the elements of  $[\dot{\tilde{q}}_2]$  and  $[\dot{\tilde{r}}_2]$  were discussed in earlier notes.

- Using **body-frame** angular velocity components gives

$$\begin{aligned}
\{0\} &= \{v_{G_2}\} - \{v_{G_1}\} - [\dot{R}_{B_1}]^T \{q'_2\} - [\dot{R}_{B_2}]^T \{r'_2\} \\
&= \{v_{G_2}\} - \{v_{G_1}\} - [R_{B_1}]^T [\tilde{\omega}'_{B_1}] \{q'_2\} - [R_{B_2}]^T [\tilde{\omega}'_{B_2}] \{r'_2\} \\
&= \{v_{G_2}\} - \{v_{G_1}\} + [R_{B_1}]^T [\tilde{q}'_2] \{\omega'_{B_1}\} + [R_{B_2}]^T [\tilde{r}'_2] \{\omega'_{B_2}\}
\end{aligned}$$

Differentiating again gives

$$\begin{aligned}
\{0\} &= \{a_{G_2}\} - \{a_{G_1}\} + [R_{B_1}]^T [\tilde{q}'_2] \{\dot{\omega}'_{B_1}\} + [\dot{R}_{B_1}]^T [\tilde{q}'_2] \{\omega'_{B_1}\} \\
&\quad + [R_{B_2}]^T [\tilde{r}'_2] \{\dot{\omega}'_{B_2}\} + [\dot{R}_{B_2}]^T [\tilde{r}'_2] \{\omega'_{B_2}\}
\end{aligned}$$

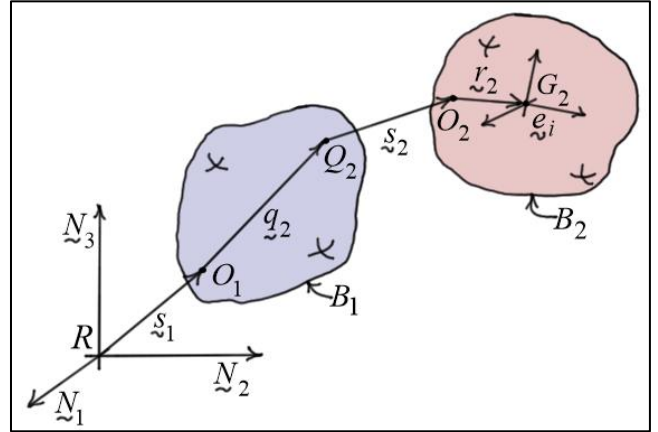
or

$$\boxed{\{0\} = \{a_{G_2}\} - \{a_{G_1}\} + [R_{B_1}]^T [\tilde{q}'_2] \{\dot{\omega}'_{B_1}\} + [R_{B_1}]^T [\tilde{\omega}'_{B_1}] [\tilde{q}'_2] \{\omega'_{B_1}\} + [R_{B_2}]^T [\tilde{r}'_2] \{\dot{\omega}'_{B_2}\} + [R_{B_2}]^T [\tilde{\omega}'_{B_2}] [\tilde{r}'_2] \{\omega'_{B_2}\}} \quad (3)$$

## Spherical Joint: Relative Coordinates

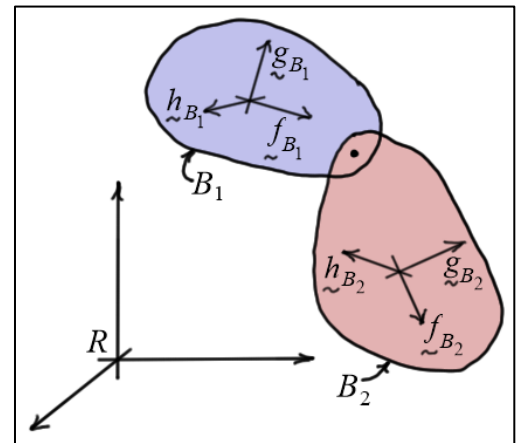
- Consider now the use of *relative coordinates* to describe the positions of points within the multibody system as shown in the figure.
- To define a *spherical joint* that attaches the points  $Q_2$  and  $O_2$ , the constraint equations are simply

$$\boxed{\{s'_2\} = \{\ddot{s}'_2\} = \{\ddot{\ddot{s}}'_2\} = \{0\}} \quad (4)$$



## Hinge (Revolute) Joint: Absolute Coordinates

- Like the spherical joint, the hinge joint connects two bodies at a single point, so the translational constraints are as given in Eq. (1).
- In addition, the hinge joint also *restricts* the *relative rotational motion* of the bodies by *eliminating* two of the three rotational degrees of freedom.
- Consider the two bodies shown in the diagram. Let  $h_{B_i}$  be a vector fixed in body  $B_i$  *parallel* to the hinge joint, and let  $f_{B_i}$  and  $g_{B_i}$  be vectors fixed in  $B_i$  that are *perpendicular* to the hinge axis (and, hence,  $h_{B_i}$ ). The *rotational constraint* can then be expressed directly in terms of the angular velocities as follows



$$\boxed{f_{B_1} \cdot {}^{B_1}\omega_{B_2} = f_{B_1} \cdot ({}^R\omega_{B_2} - {}^R\omega_{B_1}) = 0} \quad \text{or} \quad \boxed{f_{B_2} \cdot {}^{B_1}\omega_{B_2} = f_{B_2} \cdot ({}^R\omega_{B_2} - {}^R\omega_{B_1}) = 0} \quad (5)$$

$$\boxed{g_{B_1} \cdot {}^{B_1}\omega_{B_2} = g_{B_1} \cdot ({}^R\omega_{B_2} - {}^R\omega_{B_1}) = 0} \quad \text{or} \quad \boxed{g_{B_2} \cdot {}^{B_1}\omega_{B_2} = g_{B_2} \cdot ({}^R\omega_{B_2} - {}^R\omega_{B_1}) = 0} \quad (6)$$

- *Initial conditions* are used to ensure the alignment of  $h_{B_2}$  with  $h_{B_1}$ .
- Using *fixed-frame* angular velocity components, the first of Eqs. (5) and (6) can be written as follows.

$$\begin{aligned}
0 &= \{f_{B_1}'\}^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right) = \left( [R_{B_1}]^T \{f_{B_1}'\} \right)^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right) \\
&= \{f_{B_1}'\}^T [R_{B_1}] \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)
\end{aligned}$$

and

$$0 = \{g_{B_1}'\}^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right) = \{g_{B_1}'\}^T [R_{B_1}] \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)$$

- These equations can be ***differentiated*** to give

$$\begin{aligned}
0 &= \{f_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{f_{B_1}'\}^T [\dot{R}_{B_1}] \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right) \\
&= \{f_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{f_{B_1}'\}^T [R_{B_1}] [\tilde{\omega}_{B_1}]^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \{g_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{g_{B_1}'\}^T [\dot{R}_{B_1}] \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right) \\
&= \{g_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{g_{B_1}'\}^T [R_{B_1}] [\tilde{\omega}_{B_1}]^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)
\end{aligned}$$

So, finally, the two differentiated constraint equations are

$$\boxed{0 = \{f_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{f_{B_1}'\}^T [R_{B_1}] [\tilde{\omega}_{B_1}]^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)} \quad (7)$$

$$\boxed{0 = \{g_{B_1}'\}^T [R_{B_1}] \left( \{\dot{\omega}_{B_2}\} - \{\dot{\omega}_{B_1}\} \right) + \{g_{B_1}'\}^T [R_{B_1}] [\tilde{\omega}_{B_1}]^T \left( \{\omega_{B_2}\} - \{\omega_{B_1}\} \right)} \quad (8)$$

- Using ***body-frame*** angular velocity components gives

$$0 = \{f_{B_1}'\}^T \left( [{}^{B_1}R_{B_2}]^T \{\omega'_{B_2}\} - \{\omega'_{B_1}\} \right) = \{f_{B_1}'\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\omega'_{B_2}\} - \{\omega'_{B_1}\} \right)$$

$$0 = \{g_{B_1}'\}^T \left( [{}^{B_1}R_{B_2}]^T \{\omega'_{B_2}\} - \{\omega'_{B_1}\} \right) = \{g_{B_1}'\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\omega'_{B_2}\} - \{\omega'_{B_1}\} \right)$$

- These equations can be ***differentiated*** to give

$$\begin{aligned}
0 &= \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\dot{\omega}'_{B_2}\} - \{\dot{\omega}'_{B_1}\} \right) \\
&\quad + \{f'_{B_1}\}^T \left( [\dot{R}_{B_1}] [R_{B_2}]^T + [R_{B_1}] [\dot{R}_{B_2}]^T \right) \{\omega'_{B_2}\} \\
&= \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\dot{\omega}'_{B_2}\} - \{\dot{\omega}'_{B_1}\} \right) \\
&\quad + \{f'_{B_1}\}^T \left( [\tilde{\omega}'_{B_1}]^T [R_{B_1}] [R_{B_2}]^T + [R_{B_1}] [R_{B_2}]^T [\tilde{\omega}'_{B_2}] \right) \{\omega'_{B_2}\} \\
&= \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\dot{\omega}'_{B_2}\} - \{\dot{\omega}'_{B_1}\} \right) \\
&\quad + \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T [\tilde{\omega}'_{B_2}] - [\tilde{\omega}'_{B_1}] [R_{B_1}] [R_{B_2}]^T \right) \{\omega'_{B_2}\} \\
\boxed{0} &= \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\dot{\omega}'_{B_2}\} - \{\dot{\omega}'_{B_1}\} \right) \\
&\quad + \{f'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T [\tilde{\omega}'_{B_2}] - [\tilde{\omega}'_{B_1}] [R_{B_1}] [R_{B_2}]^T \right) \{\omega'_{B_2}\}
\end{aligned} \tag{9}$$

and similarly,

$$\begin{aligned}
0 &= \{g'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T \{\dot{\omega}'_{B_2}\} - \{\dot{\omega}'_{B_1}\} \right) \\
&\quad + \{g'_{B_1}\}^T \left( [R_{B_1}] [R_{B_2}]^T [\tilde{\omega}'_{B_2}] - [\tilde{\omega}'_{B_1}] [R_{B_1}] [R_{B_2}]^T \right) \{\omega'_{B_2}\}
\end{aligned} \tag{10}$$

### **Hinge (Revolute) Joint: Relative Coordinates**

- The translation constraints are the same as the spherical joint as given in Eq. (4).
- To derive the additional *rotational constraint equations*, start with the same set-up used for absolute coordinates. That is, let  $\underline{h}_{B_i}$  be a vector fixed in body  $B_i$  *parallel* to the hinge joint, and let  $\underline{f}_{B_i}$  and  $\underline{g}_{B_i}$  be vectors fixed in  $B_i$  that are *perpendicular* to the hinge joint (and, hence,  $\underline{h}_{B_i}$ ). Then, the rotational constraints can be expressed directly in terms of the angular velocities as follows

$$\boxed{\underline{f}_{B_1} \cdot {}^{B_1}\omega_{B_2} = \underline{f}_{B_1} \cdot \hat{\omega}_{B_2} = 0} \quad \text{or} \quad \boxed{\underline{f}_{B_2} \cdot {}^{B_1}\omega_{B_2} = \underline{f}_{B_2} \cdot \hat{\omega}_{B_2} = 0} \tag{11}$$

$$\boxed{\underline{\tilde{g}}_{B_1} \cdot {}^{B_1} \underline{\omega}_{B_2} = \underline{\tilde{g}}_{B_1} \cdot \hat{\underline{\omega}}_{B_2} = 0} \quad \text{or} \quad \boxed{\underline{\tilde{g}}_{B_2} \cdot {}^{B_1} \underline{\omega}_{B_2} = \underline{\tilde{g}}_{B_2} \cdot \hat{\underline{\omega}}_{B_2} = 0} \quad (12)$$

- As before, *initial conditions* are used to ensure the *alignment* of  $\underline{h}_{B_2}$  with  $\underline{h}_{B_1}$ .
- *Differentiating* the above equations and using  $B_1$  components of  $\hat{\underline{\omega}}_{B_2}$  gives

$$\boxed{\left\{ \underline{f}'_{B_1} \right\}^T \left\{ \dot{\hat{\underline{\omega}}}_{B_2} \right\} = 0} \quad (13)$$

$$\boxed{\left\{ \underline{g}'_{B_1} \right\}^T \left\{ \dot{\hat{\underline{\omega}}}_{B_2} \right\} = 0} \quad (14)$$

- *Differentiating* the above equations and using  $B_2$  components of  $\hat{\underline{\omega}}_{B_2}$  gives

$$\boxed{\left\{ \underline{f}'_{B_2} \right\}^T \left\{ \dot{\hat{\underline{\omega}}}'_{B_2} \right\} = 0} \quad (15)$$

$$\boxed{\left\{ \underline{g}'_{B_2} \right\}^T \left\{ \dot{\hat{\underline{\omega}}}'_{B_2} \right\} = 0} \quad (16)$$