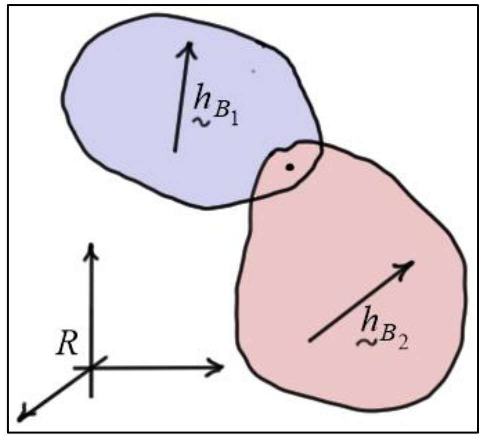
Multibody Dynamics
Connecting Joints – Part II

Two-Angle (Universal) Joint: Absolute Coordinates



- A ***universal joint*** is one that allows two bodies to share a ***common point*** (requiring three translation constraints), and it allows them to have ***two relative rotational degrees of freedom*** (requiring a single constraint on rotation).
- The three ***constraints on translation*** are the ***same*** as those given for a ***spherical joint***.
- To formulate the ***constraint on rotation***, consider the two bodies shown in the figure. Let $\underset{\sim}{h}_{B_1}$ be a vector fixed in body B_1 that describes the ***direction*** of the ***first rotation***, and let $\underset{\sim}{h}_{B_2}$ be a vector fixed in body B_2 that describes the ***direction*** of the ***second rotation***.
- The ***constraint*** can then be written

$$\boxed{\left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right) \cdot {}^{B_1}\underset{\sim}{\omega}_{B_2} = 0} \quad \text{or} \quad \boxed{\left(\underset{\sim}{h}_{B_2} \times \underset{\sim}{h}_{B_1}\right) \cdot {}^{B_1}\underset{\sim}{\omega}_{B_2} = 0} \tag{1}$$

- Using ***inertial components*** of the ***angular velocities***, Eqs. (1) can be rewritten as

$$0 = \left(\left[\tilde{h}_{B_1}\right]\left[R_{B_2}\right]^T\left\{h'_{B_2}\right\}\right)^T\left(\left\{\omega_{B_2}\right\}-\left\{\omega_{B_1}\right\}\right) = \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[\tilde{h}_{B_1}\right]^T\left(\left\{\omega_{B_2}\right\}-\left\{\omega_{B_1}\right\}\right)$$
$$= \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[\tilde{h}_{B_1}\right]\left(\left\{\omega_{B_1}\right\}-\left\{\omega_{B_2}\right\}\right)$$

The last equation can be ***differentiated*** to give

$$0 = \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[\tilde{h}_{B_1}\right]\left(\left\{\dot{\omega}_{B_1}\right\}-\left\{\dot{\omega}_{B_2}\right\}\right)$$
$$+\left\{h'_{B_2}\right\}^T\left(\left[R_{B_2}\right]\left[\dot{\tilde{h}}_{B_1}\right]+\left[\dot{R}_{B_2}\right]\left[\tilde{h}_{B_1}\right]\right)\left(\left\{\omega_{B_1}\right\}-\left\{\omega_{B_2}\right\}\right)$$

or

$$\boxed{\begin{aligned}0 &= \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[\tilde{h}_{B_1}\right]\left(\left\{\dot{\omega}_{B_1}\right\}-\left\{\dot{\omega}_{B_2}\right\}\right)\\ &\quad+\left\{h'_{B_2}\right\}^T\left(\left[R_{B_2}\right]\left[\dot{\tilde{h}}_{B_1}\right]-\left[R_{B_2}\right]\left[\tilde{\omega}_{B_2}\right]\left[\tilde{h}_{B_1}\right]\right)\left(\left\{\omega_{B_1}\right\}-\left\{\omega_{B_2}\right\}\right)\end{aligned}} \tag{2}$$

The elements of $\left[\dot{\tilde{h}}_{B_1}\right]$ are found by using the vector components of

$$\left\{\dot{h}_{B_1}\right\} = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\left\{h'_{B_1}\right\}$$

- Using ***body-fixed components*** of the ***angular velocities***, it is more convenient to write the constraint equation as

$$\begin{aligned}
0 &= \left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right)\cdot\left(\underset{\sim}{\omega}_{B_2} - \underset{\sim}{\omega}_{B_1}\right) = \left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right)\cdot \underset{\sim}{\omega}_{B_2} - \left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right)\cdot \underset{\sim}{\omega}_{B_1} \\
&= \left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right)\cdot \underset{\sim}{\omega}_{B_2} + \left(\underset{\sim}{h}_{B_2} \times \underset{\sim}{h}_{B_1}\right)\cdot \underset{\sim}{\omega}_{B_1} \\
&= \underset{\sim}{h}_{B_1}\cdot\left(\underset{\sim}{h}_{B_2} \times \underset{\sim}{\omega}_{B_2}\right) + \underset{\sim}{h}_{B_2}\cdot\left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{\omega}_{B_1}\right)
\end{aligned}$$

or, in matrix form,

$$\boxed{\{0\} = \left\{h'_{B_1}\right\}^T\left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{h}'_{B_2}\right]\left\{\omega'_{B_2}\right\} + \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{h}'_{B_1}\right]\left\{\omega'_{B_1}\right\}} \qquad (3)$$

- This result can be ***differentiated*** to put the constraint in terms of the angular acceleration components.

$$\begin{aligned}
\{0\} &= \left\{h'_{B_1}\right\}^T\left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{h}'_{B_2}\right]\left\{\dot{\omega}'_{B_2}\right\} + \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{h}'_{B_1}\right]\left\{\dot{\omega}'_{B_1}\right\} \\
&\quad + \left\{h'_{B_1}\right\}^T\left(\left[\dot{R}_{B_1}\right]\left[R_{B_2}\right]^T + \left[R_{B_1}\right]\left[\dot{R}_{B_2}\right]^T\right)\left[\tilde{h}'_{B_2}\right]\left\{\omega'_{B_2}\right\} \\
&\quad + \left\{h'_{B_2}\right\}^T\left(\left[\dot{R}_{B_2}\right]\left[R_{B_1}\right]^T + \left[R_{B_2}\right]\left[\dot{R}_{B_1}\right]^T\right)\left[\tilde{h}'_{B_1}\right]\left\{\omega'_{B_1}\right\} \\
&= \left\{h'_{B_1}\right\}^T\left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{h}'_{B_2}\right]\left\{\dot{\omega}'_{B_2}\right\} + \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{h}'_{B_1}\right]\left\{\dot{\omega}'_{B_1}\right\} \\
&\quad + \left\{h'_{B_1}\right\}^T\left(\left[\tilde{\omega}'_{B_1}\right]^T\left[R_{B_1}\right]\left[R_{B_2}\right]^T + \left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\right)\left[\tilde{h}'_{B_2}\right]\left\{\omega'_{B_2}\right\} \\
&\quad + \left\{h'_{B_2}\right\}^T\left(\left[\tilde{\omega}'_{B_2}\right]^T\left[R_{B_2}\right]\left[R_{B_1}\right]^T + \left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]\right)\left[\tilde{h}'_{B_1}\right]\left\{\omega'_{B_1}\right\}
\end{aligned} \qquad (4)$$

or

$$\boxed{\begin{aligned}
\{0\} &= \left\{h'_{B_1}\right\}^T\left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{h}'_{B_2}\right]\left\{\dot{\omega}'_{B_2}\right\} + \left\{h'_{B_2}\right\}^T\left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{h}'_{B_1}\right]\left\{\dot{\omega}'_{B_1}\right\} \\
&\quad + \left\{h'_{B_1}\right\}^T\left(\left(\left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]\right)^T + \left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\right)\left[\tilde{h}'_{B_2}\right]\left\{\omega'_{B_2}\right\} \\
&\quad + \left\{h'_{B_2}\right\}^T\left(\left(\left[R_{B_1}\right]\left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\right)^T + \left[R_{B_2}\right]\left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]\right)\left[\tilde{h}'_{B_1}\right]\left\{\omega'_{B_1}\right\}
\end{aligned}} \qquad (5)$$

Two-Angle (Universal) Joint: Relative Coordinates

- Using ***relative coordinates*** and B_1 ***components*** of ${}^{B_1}\underset{\sim}{\omega}_{B_2}$, the constraint equation

$$\boxed{0 = \left(\underset{\sim}{h}_{B_2} \times \underset{\sim}{h}_{B_1}\right) \cdot {}^{B_1}\underset{\sim}{\omega}_{B_2} = \underset{\sim}{h}_{B_2} \cdot \left(\underset{\sim}{h}_{B_1} \times {}^{B_1}\underset{\sim}{\omega}_{B_2}\right)} \qquad (6)$$

or, in matrix-vector form

$$\boxed{\left\{h'_{B_2}\right\}^T \left[R_{B_2}\right]\left[R_{B_1}\right]^T \left[\tilde{h}'_{B_1}\right]\left\{\hat{\omega}_{B_2}\right\} = \{0\}} \qquad (7)$$

- Using the B_2 ***components*** of ${}^{B_1}\underset{\sim}{\omega}_{B_2}$, the constraint equation can be written as

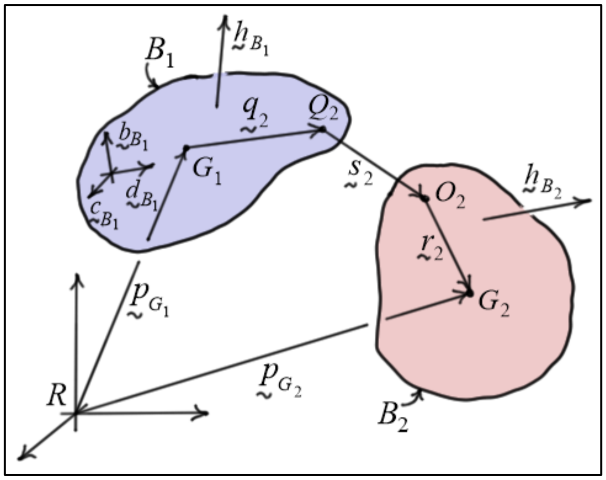
$$\boxed{0 = \left(\underset{\sim}{h}_{B_1} \times \underset{\sim}{h}_{B_2}\right) \cdot {}^{B_1}\underset{\sim}{\omega}_{B_2} = \underset{\sim}{h}_{B_1} \cdot \left(\underset{\sim}{h}_{B_2} \times {}^{B_1}\underset{\sim}{\omega}_{B_2}\right)}$$

or, in matrix-vector form

$$\boxed{\left\{h'_{B_1}\right\}^T \left[R_{B_1}\right]\left[R_{B_2}\right]^T \left[\tilde{h}'_{B_2}\right]\left\{\hat{\omega}'_{B_2}\right\} = \{0\}} \qquad (8)$$

- Eqs. (7) and (8) can be ***differentiated*** to express the constraints in terms of the angular acceleration components.

Two-Angle Joint with One Translational Degree-of-Freedom



- The ***rotational constraints*** for this joint are the ***same as*** for the ***universal joint*** described above.
- Given only one translational degree of freedom, ***two constraints*** of ***translation*** must be written.
- Consider the ***two bodies*** shown. Let $\underset{\sim}{d}_{B_1}$ be in the ***direction*** of the ***displacement*** of O_2 relative to Q_2, ***and*** let $\underset{\sim}{b}_{B_1}$ and $\underset{\sim}{c}_{B_1}$ be ***perpendicular*** to $\underset{\sim}{d}_{B_1}$.
- The ***constraints*** on ***translation*** can then be written as follows

$$\boxed{\underset{\sim}{b}_{B_1} \cdot \underset{\sim}{s}_2 = 0} \qquad \boxed{\underset{\sim}{c}_{B_1} \cdot \underset{\sim}{s}_2 = 0} \qquad (9)$$

- Using ***absolute coordinates***, the constraint equations can be written

$$\underset{\sim}{b}_{B_1} \cdot \left(\underset{\sim}{p}_{G_2} - \underset{\sim}{p}_{G_1} - \underset{\sim}{q}_2 - \underset{\sim}{r}_2 \right) = 0 \qquad \underset{\sim}{c}_{B_1} \cdot \left(\underset{\sim}{p}_{G_2} - \underset{\sim}{p}_{G_1} - \underset{\sim}{q}_2 - \underset{\sim}{r}_2 \right) = 0$$

Or, in matrix-vector form

$$\left\{ b'_{B_1} \right\}^T \left[R_{B_1} \right] \left(\left\{ p_{G_2} \right\} - \left\{ p_{G_1} \right\} - \left[R_{B_1} \right]^T \left\{ q'_2 \right\} - \left[R_{B_2} \right]^T \left\{ r'_2 \right\} \right) = 0 \tag{10}$$

$$\left\{ c'_{B_1} \right\}^T \left[R_{B_1} \right] \left(\left\{ p_{G_2} \right\} - \left\{ p_{G_1} \right\} - \left[R_{B_1} \right]^T \left\{ q'_2 \right\} - \left[R_{B_2} \right]^T \left\{ r'_2 \right\} \right) = 0 \tag{11}$$

- Using ***relative coordinates***, the matrix-vector forms of the constraint equations are

$$\left\{ b'_{B_1} \right\}^T \left\{ s'_2 \right\} = 0 \tag{12}$$

$$\left\{ c'_{B_1} \right\}^T \left\{ s'_2 \right\} = 0 \tag{13}$$

- Eqs. (10)-(13) can be differentiated to put them into the form of second order differential equations.