Multibody Dynamics
Matrices and Second Order Dyadics

Dyads and Dyadics

- A ***dyad*** is a ***vector-vector product*** that has the following properties

$(\underset{\sim}{a}\,\underset{\sim}{b})\cdot\underset{\sim}{c}=(\underset{\sim}{b}\cdot\underset{\sim}{c})\underset{\sim}{a}$	(a vector parallel to $\underset{\sim}{a}$)
$\underset{\sim}{c}\cdot(\underset{\sim}{a}\,\underset{\sim}{b})=(\underset{\sim}{c}\cdot\underset{\sim}{a})\underset{\sim}{b}$	(a vector parallel to $\underset{\sim}{b}$)
$(\underset{\sim}{a}\,\underset{\sim}{b}+\underset{\sim}{c}\,\underset{\sim}{d})\cdot\underset{\sim}{e}=(\underset{\sim}{b}\cdot\underset{\sim}{e})\underset{\sim}{a}+(\underset{\sim}{d}\cdot\underset{\sim}{e})\underset{\sim}{c}$	(a vector with components along $\underset{\sim}{a}$ and $\underset{\sim}{c}$)
$\underset{\sim}{e}\cdot(\underset{\sim}{a}\,\underset{\sim}{b}+\underset{\sim}{c}\,\underset{\sim}{d})=(\underset{\sim}{e}\cdot\underset{\sim}{a})\underset{\sim}{b}+(\underset{\sim}{e}\cdot\underset{\sim}{c})\underset{\sim}{d}$	(a vector with components along $\underset{\sim}{b}$ and $\underset{\sim}{d}$)

- ***Dyadics*** are ***linear combinations*** of dyads. A common example is the ***inertia dyadic***.
- The ***inertia dyadic*** of a body about a set of ***body-fixed*** axes passing through its ***mass-center*** G can written as

$$\boxed{\underset{\approx}{I}_G=\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}'^{G}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j} \tag{1}$$

- Here, each of the ***unit vector*** products $\underset{\sim}{e}_i\,\underset{\sim}{e}_j$ $(i,j=1,2,3)$ are called ***dyads***.
- The ***inertia values*** $I_{ij}'^{G}$ form the elements of the ***inertia matrix*** and are called the ***components*** of the ***dyadic*** in the body-fixed reference frame $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$.
- Like vectors, dyadics can be represented by ***different components*** in ***different reference frames***. Consider the dyadic $\underset{\approx}{A}$ and its representations in two different reference frames $R:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$ and $S:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$

$$\boxed{\underset{\approx}{A}=\sum_{k,\ell} a_{k\ell}^{R}\,\underset{\sim}{n}_k\,\underset{\sim}{n}_\ell=\sum_{i,j} a_{ij}^{S}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j} \tag{2}$$

- Here, $a_{k\ell}^{R}$ $(k,\ell=1,2,3)$ represent the ***components*** of $\underset{\approx}{A}$ in $R:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$, and a_{ij}^{S} $(i,j=1,2,3)$ represent the ***components*** in $S:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$.

Relationship between Dyadic Components in Different Frames

- This section formulates a ***relationship between any two sets of components*** of a dyadic. In the development of that relationship, it is assumed that the matrix $[R]^T$ transforms vectors and their components from frame $S:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ into frame $R:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$.
- The components of $\underset{\approx}{A}$ in ***two different reference frames*** can be ***related*** by noting

$$\begin{aligned}\sum_{i,j} a_{ij}^S \underset{\sim}{e}_i \underset{\sim}{e}_j &= \sum_{i,j} a_{ij}^S \left(\sum_k R_{ik}\, \underset{\sim}{n}_k\right)\left(\sum_\ell R_{j\ell}\, \underset{\sim}{n}_\ell\right) \\ &= \sum_{k,\ell}\left(\sum_{i,j}\left(a_{ij}^S R_{ik} R_{j\ell}\right)\right) \underset{\sim}{n}_k \underset{\sim}{n}_\ell \\ &= \sum_{k,\ell} a_{k\ell}^R \underset{\sim}{n}_k \underset{\sim}{n}_\ell\end{aligned}$$

- ***Comparing*** the ***last two equations***, we note that

$$\boxed{a_{k\ell}^R = \sum_{i,j} a_{ij}^S R_{ik} R_{j\ell} = \sum_{i,j} R_{ki}^T a_{ij}^S R_{j\ell}}$$

Or, in matrix form

$$\boxed{[A_R] = [R]^T [A_S][R]} \qquad (3)$$

- This result can be ***applied*** to the ***inertia matrix*** of rigid bodies. Given $[I'_G]$ the inertia matrix of a body about a set of ***body-fixed axes*** passing through the ***mass-center*** *G*, we can calculate the inertia matrix $[I''_G]$ about any other set of axes passing through *G* by using Eq. (3).

$$\boxed{[I''_G] = [R]^T [I'_G][R]}$$

- Here $[R]^T$ represents the transformation matrix that converts vector components in the "***prime***" system to vector components in the "***double prime***" system.