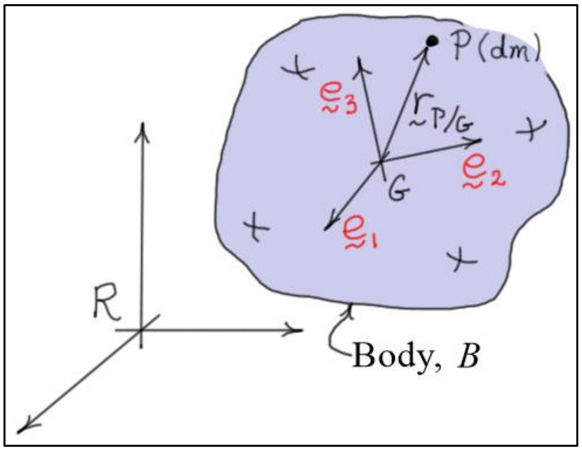
**Multibody Dynamics**
**Moments and Products of Inertia and the Inertia Matrix**

Moments of Inertia

- Consider the rigid body $B$. The ***unit vectors*** $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ are ***fixed*** in the body and directed along a ***convenient*** set of axes ***passing through the mass center G***.
- The ***moments of inertia*** of the body about these axes are defined as




$$\boxed{I_{xx}^{G} = \int_B (y^2 + z^2)\, dm}$$

$$\boxed{I_{yy}^{G} = \int_B (x^2 + z^2)\, dm}$$

$$\boxed{I_{zz}^{G} = \int_B (x^2 + y^2)\, dm}$$

- Here, $x$, $y$, and $z$ are defined as the $\underset{\sim}{e}_i$ components of $\underset{\sim}{r}_{P/G}$ the position vector of a typical point $P$ with respect to $G$, that is, $\boxed{\underset{\sim}{r}_{P/G} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3}$.
- ***Moments of inertia*** of a body about an axis are a ***measure*** of the ***distribution*** of the ***body's mass about that axis***. The ***smaller*** the ***inertia*** the more the mass is ***concentrated*** about the axis.
- Inertia values can be found either by ***measurement*** or ***calculation***. Calculations are based on direct integration or on the "***body build-up***" technique. In the body build-up technique, inertias of ***simple shapes*** are ***added*** (or subtracted) to estimate the inertia of a composite shape.
- The inertias of ***simple shapes*** (about their individual mass centers) are found in ***standard inertia tables***. These values are ***transferred*** to axes through the composite mass center using the ***Parallel Axes Theorem*** for ***Moments of Inertia***.

Parallel Axes Theorem for Moments of Inertia

- The inertia $(I_i^A)$ of a body about an axis $(i)$ through any point $(A)$ is equal to the inertia $(I_i^G)$ of the body about a parallel axis through the mass center $G$ plus the mass $(m)$ times the distance $(d_i)$ between the two parallel axes squared.

$$\boxed{I_{ii}^{A} = I_{ii}^{G} + m\,d_i^2} \quad (i = x, y, \text{ or } z)$$

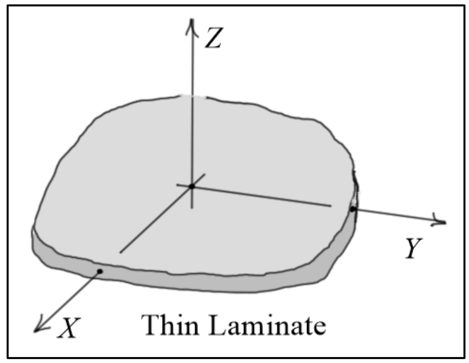
- ***Moments of inertia*** are ***always positive***. From the parallel axes theorem, it is obvious that the ***minimum moments of inertia*** of a body occur about axes passing through its ***mass center***.

Products of Inertia

- ***Products of inertia*** of the rigid body are defined as

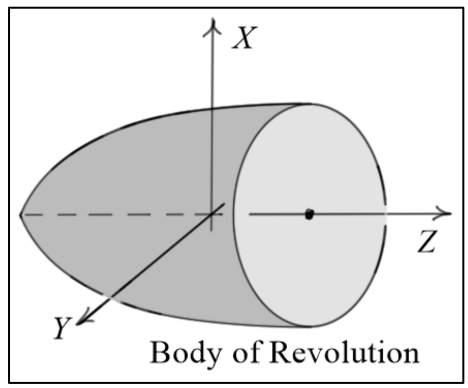
$$\boxed{I_{xy}^{G} = \int_B (xy)\,dm} \quad \boxed{I_{xz}^{G} = \int_B (xz)\,dm}, \quad \boxed{I_{yz}^{G} = \int_B (yz)\,dm}$$

- Products of inertia of a body are ***measures*** of ***body symmetry***.
- If a plane is a ***plane of symmetry***, the products of inertia associated with any axis ***perpendicular*** to that plane are ***zero***.
- For example, consider a ***thin laminate***. The mid-plane of the laminate lies in the *XY*-plane so that half its thickness is above the plane and half is below. Hence, the *XY*-plane is a plane of symmetry and




$$\boxed{I_{xz} = I_{yz} = 0}$$

- ***Bodies of revolution*** have ***two planes of symmetry***. For the configuration shown, the *XZ* and *YZ* planes are planes of symmetry. Hence, ***all products*** of inertia ***are zero*** about the *X*, *Y*, and *Z* axes.




- Products of inertia are found either by ***measurement*** or ***calculation***. Calculations are based on direct integration or on the "***body build-up***" technique. In the body build-up technique, products of inertia of ***simple shapes*** are ***added*** (or subtracted) to estimate the products of inertia of a ***composite shape***.
- The products of inertia of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are ***transferred*** to axes through the composite mass center using the ***Parallel Axes Theorem*** for ***Products of Inertia***.

Parallel Axes Theorem for Products of Inertia

- The product of inertia $(I_{ij}^{A})$ of a body about a pair of axes $(i,j)$ passing through any point $(A)$ is equal to the product of inertia $(I_{ij}^{G})$ of the body about a set of parallel axes through the mass center $G$ plus the mass $(m)$ times the product of the coordinates $(c_i c_j)$ of $G$ relative to $A$ measured along those axes.

$$\boxed{I_{ij}^{A} = I_{ij}^{G} + m c_i c_j} \qquad (i, j = x, y, \text{ or } z)$$

- ***Products of inertia*** can be ***positive***, ***negative***, or ***zero***.

The Inertia Matrix

- The inertias of a body about a set of axes (passing through some point) are collected into a single inertia matrix. For example, the inertia matrix of a body about a set of axes through its mass center $G$ is defined as

$$\boxed{\left[I_G\right] = \begin{bmatrix} I_{11}^{G} & I_{12}^{G} & I_{13}^{G} \\ I_{21}^{G} & I_{22}^{G} & I_{23}^{G} \\ I_{31}^{G} & I_{32}^{G} & I_{33}^{G} \end{bmatrix} = \begin{bmatrix} I_{xx}^{G} & -I_{xy}^{G} & -I_{xz}^{G} \\ -I_{xy}^{G} & I_{yy}^{G} & -I_{yz}^{G} \\ -I_{xz}^{G} & -I_{yz}^{G} & I_{zz}^{G} \end{bmatrix}}$$

- There is a ***different inertia matrix*** for ***each set of axes*** passing through a given point.
- There is one set of directions for each point that renders the inertia matrix ***diagonal***. These directions are called ***principal directions*** (or ***principal axes***) of the body for that point. In general, the ***principal axes*** are different for each point in a body.
- ***All*** inertia matrices are ***symmetric***.