

Multibody Dynamics

Principal Moments of Inertia and Principal Directions

Properties of Real Symmetric Matrices

(Reference: Richard Bronson, *Matrix Methods: An Introduction*, Academic Press, 1970.)

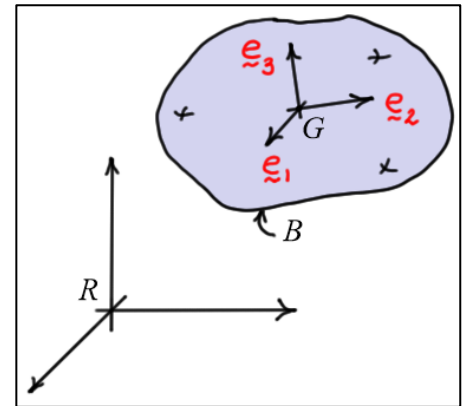
All *real, symmetric matrices* have the following *properties*:

- The *eigenvalues* of a real, symmetric matrix are *real*.
- The *eigenvectors* of a real, symmetric matrix can always be chosen to be *real*.
- A real, symmetric matrix is *diagonalizable*.
- Every real, symmetric matrix possesses a *complete set* of *orthonormal eigenvectors*.
- For every real symmetric $n \times n$ matrix $[A]$, there exists an $n \times n$ *real orthogonal matrix* $[M]$ such that $[M]^T[A][M]=[D]$, where $[D]$ is a diagonal matrix.
- $[M]$ is called the *modal matrix*. Its columns are formed by the *eigenvectors* of $[A]$.
- $[D]$ is a diagonal matrix whose entries are the *eigenvalues* of $[A]$.
- *Eigenvalues* appear in the *same columns* of $[D]$ as their *associated eigenvectors* appear in $[M]$.

Principal Moments of Inertia and Principal Directions

- The *inertia dyadic* of the body for a set of axes passing through its mass-center and parallel to the unit vector set $B:(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is defined as

$$\underline{\underline{I}}'_G = \sum_{i=1}^3 \sum_{j=1}^3 I'_{ij} \underline{e}_i \underline{e}_j$$



- The *components* of the inertia dyadic form a *real, symmetric* 3×3 *matrix* $[I'_G]$. Consequently, this matrix has all the properties listed above.
- The eigenvectors of $[I'_G]$ define the *principal directions* of the body for the mass-center G . Recall that the *principal directions* are those directions for which *all products of inertia* are *zero*.

- The *eigenvalues* of $[I'_G]$ are the *principal moments of inertia* of the body for G , that is, they are the *moments of inertia* about the *principal directions*.
- Note that the *principal directions* and inertia *vary* from point to point in the body, but *there exists only one set of principal directions for any given point*.

Calculation of Principal Moments of Inertia and Principal Directions

- The *principal moments of inertia* of a body for a given point, say the mass-center G , can be calculated (as you would to find the eigenvalues of any 3×3 matrix) by setting

$$\det \begin{bmatrix} (I'_{xx} - \lambda) & -I'_{xy} & -I'_{xz} \\ -I'_{xy} & (I'_{yy} - \lambda) & -I'_{yz} \\ -I'_{xz} & -I'_{yz} & (I'_{zz} - \lambda) \end{bmatrix} = 0. \quad (1)$$

By *expanding* the *determinant*, the resulting characteristic equation can be written as

$$\begin{aligned} \lambda^3 + (-I'_{xx} - I'_{yy} - I'_{zz})\lambda^2 + (I'_{xx}I'_{yy} + I'_{xx}I'_{zz} + I'_{yy}I'_{zz} - I'^2_{xy} - I'^2_{xz} - I'^2_{yz})\lambda \\ + (-I'_{xx}I'_{yy}I'_{zz} + I'^2_{xx}I'_{yz} + I'^2_{yy}I'_{xz} + I'^2_{zz}I'_{xy} + 2I'_{xy}I'_{xz}I'_{yz}) = 0 \end{aligned} \quad (2)$$

The *three roots* to this equation are the *three principal moments of inertia*.

- If I_i ($i=1,2,3$) represent the three principal moments of inertia, the principal directions can be found (as you would find the eigenvectors of any 3×3 matrix) by writing

$$\begin{bmatrix} (I'_{xx} - I_i) & -I'_{xy} & -I'_{xz} \\ -I'_{xy} & (I'_{yy} - I_i) & -I'_{yz} \\ -I'_{xz} & -I'_{yz} & (I'_{zz} - I_i) \end{bmatrix} \begin{Bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{Bmatrix} = \{0\} \quad (3)$$

- Since the coefficient matrix is *singular*, these equations *do not have a single solution*. Eigenvectors can be found to within a constant multiplier only.
- If the eigenvectors are defined to be *unit vectors*, then $a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1$. In this case, the eigenvectors are unique. Recall that the components of a unit vector are the direction cosines for that direction.
- To solve Eq. (3), simply choose a value for one of the a_{ij} ($j=1,2,3$), and then solve for the other two. Then, *normalize* the resulting vector.