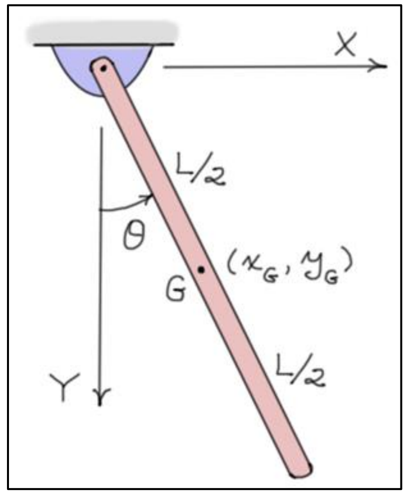
Multibody Dynamics
Constraints for Multibody Systems

Configuration (or Holonomic) Constraints



- Suppose the configuration of a multibody system is defined by "*n*" generalized coordinates, say $q_k \; (k=1,\ldots,n)$.
- These coordinates may all be ***independent***, or they may form a ***dependent set***.
- For example, consider the simple pendulum. The coordinate set $\{x_G, y_G, \theta\}$ is a ***dependent set***. The coordinates can be related using the following equations.

  $$x_G = \frac{L}{2}\sin(\theta) \qquad y_G = \frac{L}{2}\cos(\theta)$$

  Hence, for this system only ***one*** generalized coordinate is required. Any set of ***two*** or ***more*** coordinates forms a ***dependent*** set.
- The types of constraints described above are referred to as ***configuration constraints***. For a mechanical system described by "*n*" ***generalized coordinates*** $q_k \; (k=1,\ldots,n)$ with "*m*" ***configuration constraints***, the constraints can be written as

  $$\boxed{f_j(q_1, q_2, \ldots, q_n, t) = 0} \qquad (j=1,\ldots,m) \qquad (1)$$

- Here, the constraints ***may also be dependent*** on the ***time***, *t*.
- Configuration constraints are called ***holonomic*** constraints. If they ***depend explicitly on time***, they are ***rheonomic*** constraints. If they ***do not depend explicitly on time***, they are called ***schleronomic*** constraints.
- Configuration constraints are most useful in a dynamic analysis when they are ***differentiated*** into a form that is ***linear*** in the ***time derivatives*** of the ***generalized coordinates*** or ***generalized speeds***.
- Differentiating Eq. (1) gives

  $$\boxed{\frac{df_j}{dt} = \sum_{k=1}^{n}\left(\frac{\partial f_j}{\partial q_k}\right)\dot{q}_k + \frac{\partial f_j}{\partial t} = \sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m) \qquad (2)$$

Examples of Configuration Constraints

1. For the simple pendulum shown above, define the generalized coordinate set as $\{q_1, q_2, q_3\} = \{x_G, y_G, \theta\}$. The two configuration constraints are

$$\begin{Bmatrix} f_1(q_1,q_2,q_3) \\ f_2(q_1,q_2,q_3) \end{Bmatrix} = \begin{Bmatrix} q_1 - \frac{L}{2}\sin(q_3) \\ \\ q_2 - \frac{L}{2}\cos(q_3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

These equations can be differentiated to get

$$\begin{Bmatrix} \dot{q}_1 - \frac{L}{2}\dot{q}_3\cos(q_3) \\ \\ \dot{q}_2 + \frac{L}{2}\dot{q}_3\sin(q_3) \end{Bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & -\frac{L}{2}\cos(q_3) \\ \\ 0 & 1 & +\frac{L}{2}\sin(q_3) \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Comparing these results to the general form shown in Eq. (2), we conclude

$$\boxed{a_{11} = a_{22} = 1} \qquad \boxed{a_{12} = a_{21} = 0} \qquad \boxed{a_{13} = -\frac{L}{2}\cos(\theta)} \qquad \boxed{a_{23} = +\frac{L}{2}\sin(\theta)}$$

and

$$\boxed{a_{j0} = 0} \qquad (j = 1,2)$$

2. For the simple pendulum shown above, define the generalized coordinate set as $\{q_1, q_2\} = \{x_G, y_G\}$. The configuration constraint is

$$f(q_1, q_2) = q_1^2 + q_2^2 - (L/2)^2 = 0$$

and differentiating gives

$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 = 0 \qquad \text{or} \qquad \boxed{[q_1 \quad q_2]\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = 0}$$

So, in this case, $\boxed{a_{11} = q_1 = x_G}$, $\boxed{a_{12} = q_2 = y_G}$, and $\boxed{a_{10} = 0}$.

Motion Constraints

- ***Motion constraints*** will occur directly in the ***form of*** Eq. (2). It ***may*** or ***may not*** be possible to ***integrate*** the ***constraint*** to find a configuration form of the constraint as given by Eq. (1). Motion constraints that ***cannot*** be ***integrated*** are called ***nonholonomic constraints***.
- It may be ***difficult to tell***, however, whether a motion constraint is ***integrable*** or not, so it may be ***difficult*** to ***determine*** if a constraint is ***holonomic*** or ***nonholonomic***.
- For example, suppose the motion constraint is of the form of Eq. (2). If the constraint is ***holonomic***, then

$$\boxed{a_{jk} = \frac{\partial f_j}{\partial q_k}}$$

Moreover, if $f_j(q_1, q_2, \ldots, q_n, t)$ and all its derivatives are continuous,

$$\boxed{\frac{\partial a_{jk}}{\partial q_\ell} = \frac{\partial^2 f_j}{\partial q_\ell \partial q_k} = \frac{\partial^2 f_j}{\partial q_k \partial q_\ell} = \frac{\partial a_{j\ell}}{\partial q_k}} \qquad k, \ell = (1, \ldots, n) \qquad (3)$$

and

$$\boxed{\frac{\partial a_{j0}}{\partial q_\ell} = \frac{\partial^2 f_j}{\partial q_\ell \partial t} = \frac{\partial^2 f_j}{\partial t \partial q_\ell} = \frac{\partial a_{j\ell}}{\partial t}} \qquad \ell = (1, \ldots, n) \qquad (4)$$

- If, for a given motion constraint, Eqs. (3) and (4) are ***satisfied***, then the motion constraint is ***holonomic***. However, if these equations are ***not true***, we ***cannot conclude*** that the constraint is ***nonholonomic***, because it is possible that an ***integrating factor*** $g_j(q_1, q_2, \ldots, q_n, t)$ exists that transforms the equation into an ***exact differential***.
- If an integrating factor exists, then

$$\boxed{\frac{\partial\left(g_j a_{jk}\right)}{\partial q_\ell} = \frac{\partial\left(g_j a_{j\ell}\right)}{\partial q_k}} \quad k, \ell = (1, \ldots, n) \qquad \boxed{\frac{\partial\left(g_j a_{j0}\right)}{\partial q_\ell} = \frac{\partial\left(g_j a_{j\ell}\right)}{\partial t}} \quad \ell = (1, \ldots, n) \qquad (5)$$

- Unfortunately, ***finding the integrating factor is not always an easy task***.