Multibody Dynamics
Lagrange's Equations for MDOF Systems with Constraints

Background

- As discussed in earlier notes, dynamic systems may be subjected to ***holonomic*** and/or ***nonholonomic*** constraints. The nature of these constraints determines how they will be incorporated into Lagrange's equations. Consider the following three scenarios:
  1. The constraints are ***holonomic***, and it is easy to use them to ***eliminate*** the ***surplus generalized coordinates***. It is helpful here if the resulting expressions for the kinetic and potential energies of the system are ***reasonable*** to ***differentiate***.
  2. The constraints are ***holonomic***; however, ***elimination*** of surplus generalized coordinates ***leads to very complicated expressions*** for the kinetic and potential energies of the system making them difficult (very tedious) to differentiate.
  3. The constraints are ***nonholonomic***, so surplus generalized coordinates (because the constraints are not integrable) cannot be eliminated.
- In the first scenario, the surplus coordinates are eliminated and the form of ***Lagrange's equations without constraints*** is used (discussed in earlier notes).
- In the second and third scenarios, a dependent set of generalized coordinates is used which requires a form of Lagrange's equations that incorporates the constraints into the formulation of the equations of motion. This form of Lagrange's equations is presented below.

Lagrange's Equations with Constraints

- If the configuration of a dynamic system is to be described using "n" generalized coordinates $q_k \; (k=1,\ldots,n)$ and if the system is subjected to "m" ***independent*** holonomic and/or nonholonomic ***constraints*** of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m) \tag{1}$$

  the system possesses $N = n - m$ degrees of freedom (DOF).
- In this case, the equations of motion of the system can be formulated using ***Lagrange's equations*** with ***Lagrange multipliers*** as presented in Eq. (2).

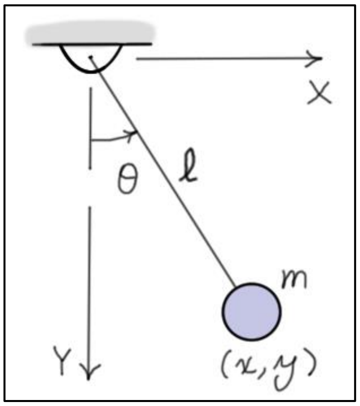
$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right)-\frac{\partial K}{\partial q_k} = F_{q_k}+\sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \qquad (2)$$

- Here, K is the ***kinetic energy*** of the system, F_{q_k} is the ***generalized force*** associated with the generalized coordinate q_k, λ_j is the ***Lagrange multiplier*** associated with the j^{th} constraint equation, and a_{jk} $(j=1,\ldots,m;\ k=1,\ldots,n)$ are the ***constraint equation coefficients***.
- If some of the forces and torques are ***conservative***, their contribution to the equations of motion can be found in terms of a ***potential energy function***. In this case, Lagrange's equations of motion can be rewritten as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc}+\sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \qquad (3)$$

- Here, $L = K - V$ is the ***Lagrangian*** of the system, V is the ***potential energy*** function for the ***conservative*** forces and torques, $\left(F_{q_k}\right)_{nc}$ is the ***generalized force*** associated with q_k for the ***nonconservative*** forces and torques, only, λ_j is the ***Lagrange multiplier*** associated with the j^{th} constraint equation, and a_{jk} $(j=1,\ldots m; k=1,\ldots n)$ are the ***constraint equation coefficients***.
- The "n" Lagrange's equations and the "m" constraint equations (1) form a set of "$n+m$" ***differential/algebraic*** equations for the "$n+m$" unknowns – the "n" generalized coordinates q_k $(k=1,\ldots,n)$ and the "m" Lagrange multipliers λ_j $(j=1,\ldots,m)$.
- The equations are ***differential*** in the ***generalized coordinates*** and ***algebraic*** in the ***Lagrange multipliers***. The Lagrange multipliers are related to the ***forces*** and ***moments*** required to ***maintain*** the ***constraints***. Note that, it is essential that K, F_{q_k}, and a_{jk} be written only in terms of q_k and $\dot{q}_k$, and no other variables.

Example: Equations of Motion of the Simple Pendulum



Approach #1: Using θ as the generalized coordinate

- The simple pendulum is a single degree of freedom (SDOF) system, and using θ as the generalized coordinate, the Lagrangian can be written as

$$\boxed{L = \tfrac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta)}$$

Using Lagrange's equations ***without constraints***, the equation of motion can be written as

$$\boxed{\ddot{\theta} + \tfrac{g}{\ell}\sin(\theta) = 0} \qquad (4)$$

Approach #2: Using (x, y) as the generalized coordinates

- Using two generalized coordinates for an SDOF system, we must use Lagrange's equations with a constraint. In this case, we can write a ***configuration constraint*** as

$$\boxed{x^2 + y^2 = \ell^2}$$

- ***Differentiating***, the constraint can be written in the form of Eq. (1) and the coefficients a_{jk} $(j = 1;\, k = 0,1,2)$ can be identified as follows.

$$\boxed{x\dot{x} + y\dot{y} = 0} \text{ and } \boxed{a_{10} = 0, \quad a_{11} = x, \quad a_{12} = y} \qquad (5)$$

- Lagrange's equations ***with constraints*** can then be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \lambda a_{1k}} \qquad (k = 1,2) \qquad (6)$$

- The Lagrangian can be written in terms of the generalized coordinates and their derivatives as

$$\boxed{L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + mgy}$$

- Using Eq. (6) and the definitions of the constraint equation coefficients in Eq. (5), the following two differential/algebraic equations of motion are found.

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x &= 0 \\ m\ddot{y} - mg - \lambda y &= 0 \end{aligned}} \qquad (7)$$

- Differentiating the constraint Eq. (5) again, and combining it with Eq. (7) gives the following ***complete set*** of ***three differential/algebraic equations*** for (x, y, λ).

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x &= 0 \\ m\ddot{y} - mg - \lambda y &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \tag{8}$$

- ***Alternatively***, the first two of the above equations can be used to ***eliminate*** the ***Lagrange multiplier*** λ. For example, by ***multiplying*** the ***first equation*** by "*y*" and the ***second equation*** by "*x*" and ***subtracting***, the first two equations can be reduced to a single equation. The result is a set of only ***two coupled differential equations of motion***

$$\boxed{\begin{aligned} y\ddot{x} - x\ddot{y} - gx &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \tag{9}$$

- The coupled equations shown in Equation (9) can be written in matrix form

$$\boxed{\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} -gx \\ -\dot{x}^2 - \dot{y}^2 \end{Bmatrix}} \tag{10}$$

- Equations (10) can be solved simultaneously to calculate the motion of the pendulum for any set of initial values. Or, the two equations can be decoupled first to simplify the numerical solution process. If the equations are solved directly, it must be solved as a set of linear algebraic equations to find $\ddot{x}$ and $\ddot{y}$ at each step of the numerical integration process.
- To decouple the two equations prior to the numerical solution, Cramer's rule can be used.

$$\boxed{\ddot{x} = \frac{\det\begin{bmatrix} f_1 & -x \\ f_2 & y \end{bmatrix}}{\det\begin{bmatrix} y & -x \\ x & y \end{bmatrix}} = \frac{(yf_1 + xf_2)}{(x^2 + y^2)}} \quad \text{and} \quad \boxed{\ddot{y} = \frac{\det\begin{bmatrix} y & f_1 \\ x & f_2 \end{bmatrix}}{\det\begin{bmatrix} y & -x \\ x & y \end{bmatrix}} = \frac{(yf_2 - xf_1)}{(x^2 + y^2)}}$$