**Multibody Dynamics**
**Constraint Relaxation Method: Meaning of Lagrange Multipliers**

- Previously, it was noted that if a dynamic system is described using "*n*" ***generalized coordinates*** $q_k\ (k=1,\ldots,n)$, and if the system is subjected to "*m*" ***independent holonomic*** and/or ***nonholonomic*** constraint equations of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m) \tag{1}$$

tthe equations of motion of the system can be found by using one of the following two forms of Lagrange's equations with Lagrange multipliers.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{2}$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{3}$$

Here,

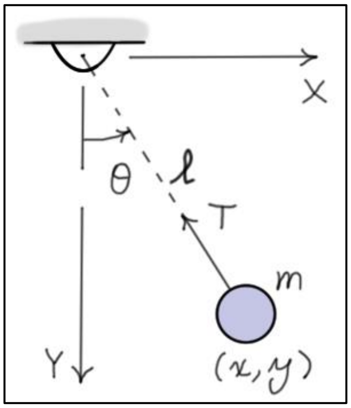
- $K$ : ***kinetic energy*** of the system
- $F_{q_k}$ : ***generalized force*** associated with the generalized coordinate $q_k$
- $L$ : ***Lagrangian*** of the system
- $V$ : ***potential energy*** function for the ***conservative*** forces and torques
- $\left(F_{q_k}\right)_{nc}$ : ***generalized force*** associated with $q_k$ for ***only*** the ***nonconservative*** forces/torques
- $\lambda_j$ : Lagrange multiplier associated with the $j^{\text{th}}$ constraint equation
- $a_{jk}$ : coefficients from the constraint equations $\left(j=1,\ldots,m;\ k=1,\ldots,n\right)$

- Eqs. (1) and (2) or Eqs. (1) and (3) form a set of "$n+m$" ***differential/algebraic equations*** for the "*n*" ***generalized coordinates*** and the "*m*" ***Lagrange multipliers***.
- Alternatively, some or all the constraints can be ***relaxed*** (or ***removed***) and replaced with ***force*** and/or ***torque*** components that are required to ***maintain*** the ***constraints***. Then, formulate the "*n*" Lagrange's equations in terms of the "*n*" generalized coordinates and the "*m*" constraint force (or torque) components.

- ***Together*** with the ***constraint equations***, this forms a set of "$n+m$" ***differential/ algebraic equations*** for the "$n$" ***generalized coordinates*** and the "$m$" ***constraint force*** and/or ***torque components***. If ***all*** the constraints are ***relaxed***, then Eqs. (3) can be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \left(F_{q_k}\right)_{\text{constraints}}} \qquad (k=1,\ldots,n) \qquad (4)$$

Example: The Simple Pendulum




- For the simple pendulum shown, $q_1 = x$ and $q_2 = y$ are used as the generalized coordinates, and the ***length constraint*** of the pendulum is ***relaxed*** in the formulation. Lagrange's equations can then be written in the form of Eq. (4).
- Here, $L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mgy$, $\left(F_{q_k}\right)_{nc}=0$, and the contributions of the constraint force to the right sides of the equations are

$$\boxed{\left(F_x\right)_{\text{constraint}} = \underset{\sim}{T}\cdot\left(\partial \underset{\sim}{v}/\partial\dot{x}\right) = T\left(-\left(x/\ell\right)\underset{\sim}{i} - \left(y/\ell\right)\underset{\sim}{j}\right)\cdot\partial\left(\dot{x}\underset{\sim}{i}+\dot{y}\underset{\sim}{j}\right)\Big/\partial\dot{x} = -T\left(x/\ell\right)} \qquad (5)$$

$$\boxed{\left(F_y\right)_{\text{constraint}} = \underset{\sim}{T}\cdot\left(\partial \underset{\sim}{v}/\partial\dot{y}\right) = T\left(-\left(x/\ell\right)\underset{\sim}{i} - \left(y/\ell\right)\underset{\sim}{j}\right)\cdot\partial\left(\dot{x}\underset{\sim}{i}+\dot{y}\underset{\sim}{j}\right)\Big/\partial\dot{y} = -T\left(y/\ell\right)} \qquad (6)$$

- Substituting into Lagrange's equations (4) and supplementing with the twice differentiated constraint equation gives the following equations of motion.

$$\boxed{\begin{aligned} m\ddot{x}+\left(\tfrac{x}{\ell}\right)T &= 0\\ m\ddot{y}-mg+\left(\tfrac{y}{\ell}\right)T &= 0\\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2 &= 0\end{aligned}} \qquad (7)$$

- Using ***Lagrange multipliers***, it was shown in previous notes that the equations for the pendulum could be written as

$$\boxed{\begin{aligned} m\ddot{x}-\lambda x &= 0\\ m\ddot{y}-mg-\lambda y &= 0\\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2 &= 0\end{aligned}} \qquad (8)$$

- Comparing Eqs. (7) and (8), it is clear that the ***Lagrange multiplier*** $\lambda = -T/\ell$, the tension force per unit pendulum length.