Multibody Dynamics
d'Alembert's Principle for Multi-Degree-of-Freedom (MDOF) Systems

- Consider a system of "N_B" rigid bodies with "n" degrees of freedom (DOF). Previously, we noted that the equations of motion (EOM) of a such a system can be written for the "n" generalized coordinates $q_k\ (k=1,\ldots,n)$ using Lagrange's equations. Another approach to finding the EOM is to use ***d'Alembert's principle***. For a system of "N_B" rigid bodies with "n" DOF, d'Alembert's principle can be written as

$$\boxed{\sum_{i=1}^{N_B}\left(m_i \underset{\sim}{a}_{G_i}\cdot\frac{\partial \underset{\sim}{v}_{G_i}}{\partial \dot{q}_k}\right)+\sum_{i=1}^{N_B}\left[\left(\underset{\approx}{I}_{G_i}\cdot\underset{\sim}{\alpha}_{B_i}\right)+\left(\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial \underset{\sim}{\omega}_{B_i}}{\partial \dot{q}_k}=F_{q_k}} \qquad (k=1,\ldots,n) \qquad (1)$$

Here,

m_{B_i} = mass of body B_i

$\underset{\sim}{v}_{G_i}$ = velocity of G_i, the mass center of B_i

$\underset{\sim}{a}_{G_i}$ = acceleration of G_i, the mass center of B_i

$\underset{\sim}{\omega}_{B_i}$ = angular velocity of B_i

$\underset{\sim}{\alpha}_{B_i}$ = angular acceleration of B_i

$\underset{\approx}{I}_{G_i}$ = inertia dyadic for B_i

$\underset{\sim}{H}_{G_i}$ = angular momentum of B_i about its mass center G_i

F_{q_k} = generalized force associated with generalized coordinate q_k
(due to all forces and torques acting on the system)

Notes

1. Eqs. (1) represent "n" differential equations for the "n" generalized coordinates $q_k\ (k=1,\ldots,n)$. It is important that all quantities be written only in terms of q_k, $\dot{q}_k$, and no other variables.
2. The ***right-hand-side*** of the EOM are the ***same*** as for Lagrange's equations. All forces and torques (both conservative and nonconservative) are included in F_{q_k}. There is no special form for conservative systems.
3. This form of d'Alembert's principle can be used for systems ***without constraints*** or for systems with ***holonomic constraints*** provided it is easy to use the constraints to ***eliminate surplus generalized coordinates***.

4. For systems with ***nonholonomic constraints*** or for systems with ***holonomic constraints*** for which it is not convenient to eliminate surplus generalized coordinates, then ***Lagrange multipliers*** are used.

<u>d'Alembert's Principle with Constraints</u>

- If the configuration of a dynamic system is to be described using "n" ***generalized coordinates*** $q_k \; (k=1,\ldots,n)$, and if the system is subjected to "m" ***independent*** holonomic and/or nonholonomic ***constraints*** of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} \; = \; 0} \qquad (j=1,\ldots,m) \tag{2}$$

  the system possesses "$N = n - m$" DOF. In this case, the EOM of the system can be formulated using d'Alembert's principle with Lagrange multipliers which can be written as follows.

$$\boxed{\sum_{i=1}^{N_B}\left(m_i \underset{\sim}{a}_{G_i} \cdot \frac{\partial \underset{\sim}{v}_{G_i}}{\partial \dot{q}_k} \right) + \sum_{i=1}^{N_B}\left[\left(\underset{\approx}{I}_{G_i} \cdot \underset{\sim}{\alpha}_{B_i} \right) + \left(\underset{\sim}{\omega}_{B_i} \times \underset{\sim}{H}_{G_i} \right)\right] \cdot \frac{\partial \underset{\sim}{\omega}_{B_i}}{\partial \dot{q}_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk}} \quad (k=1,\ldots,n) \tag{3}$$

- Here, λ_j is the ***Lagrange multiplier*** associated with the j^{th} constraint equation, $a_{jk} \; (j=1,\ldots,m; \; k=1,\ldots,n)$ are the ***coefficients*** from the constraint equations, and all other quantities are as defined above.
- The "n" d'Alembert's principle equations (3) and the "m" constraint equations (2) form a set of "$n+m$" ***differential/algebraic*** equations for the "$n+m$" unknowns – the "n" generalized coordinates $q_k \; (k=1,\ldots,n)$ and the "m" Lagrange multipliers $\lambda_j \; (j=1,\ldots,m)$.
- The equations are ***differential*** in the ***generalized coordinates*** and ***algebraic*** in the ***Lagrange multipliers***. The Lagrange multipliers are related to the forces and moments required to maintain the constraints. Note that, it is important that all quantities be written only in terms of q_k, $\dot{q}_k$, and no other variables.