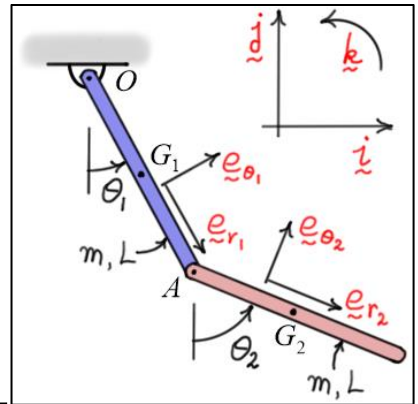
Multibody Dynamics
Examples using d'Alembert's Principle

Examples

1. **Double Pendulum**: Find the EOM of the double pendulum shown using d'Alembert's principle and θ_1 and θ_2 as the generalized coordinates. Both links have mass m and length L.



d'Alembert's Principle:

$$\boxed{\sum_{i=1}^{2}\left(m_i \underset{\sim}{a}_{G_i}\cdot\frac{\partial \underset{\sim}{v}_{G_i}}{\partial\dot{\theta}_k}\right)+\sum_{i=1}^{2}\left[\left(\underset{\approx}{I}_{G_i}\cdot\underset{\sim}{\alpha}_{B_i}\right)+\left(\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_{B_i}}{\partial\dot{\theta}_k}=F_{q_k}} \qquad (k=1,\ldots,2)$$

Here,

$$\underset{\sim}{\omega}_{B_1}=\dot{\theta}_1\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}_{B_1}\big/\partial\dot{\theta}_1=\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}_{B_1}\big/\partial\dot{\theta}_2=\underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{B_2}=\dot{\theta}_2\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}_{B_2}\big/\partial\dot{\theta}_1=\underset{\sim}{0} \qquad \partial\underset{\sim}{\omega}_{B_2}\big/\partial\dot{\theta}_2=\underset{\sim}{k}$$

$$\underset{\sim}{v}_{G_1}=\tfrac{1}{2}L\dot{\theta}_1\underset{\sim}{e}_{\theta_1} \qquad \partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_1=\tfrac{1}{2}L\underset{\sim}{e}_{\theta_1} \qquad \partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_2=0$$

$$\underset{\sim}{v}_{G_2}=L\dot{\theta}_1\underset{\sim}{e}_{\theta_1}+\tfrac{1}{2}L\dot{\theta}_2\underset{\sim}{e}_{\theta_2} \qquad \partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_1=L\underset{\sim}{e}_{\theta_1} \qquad \partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_2=\tfrac{1}{2}L\underset{\sim}{e}_{\theta_2}$$

$$\underset{\sim}{a}_{G_1}=\tfrac{1}{2}L\ddot{\theta}_1\underset{\sim}{e}_{\theta_1}-\tfrac{1}{2}L\dot{\theta}_1^2\underset{\sim}{e}_{r_1} \qquad \underset{\sim}{a}_{G_2}=\left(L\ddot{\theta}_1\underset{\sim}{e}_{\theta_1}-L\dot{\theta}_1^2\underset{\sim}{e}_{r_1}\right)+\left(\tfrac{1}{2}L\ddot{\theta}_2\underset{\sim}{e}_{\theta_2}-\tfrac{1}{2}L\dot{\theta}_2^2\underset{\sim}{e}_{r_2}\right)$$

$$m_1\underset{\sim}{a}_{G_1}\cdot\left(\partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_1\right)=\tfrac{1}{4}mL^2\ddot{\theta}_1 \qquad m_1\underset{\sim}{a}_{G_1}\cdot\left(\partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_2\right)=0$$

$$m_2\underset{\sim}{a}_{G_2}\cdot\left(\partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_1\right)=m\left(L^2\ddot{\theta}_1+\tfrac{1}{2}L^2\ddot{\theta}_2C_{2-1}-\tfrac{1}{2}L^2\dot{\theta}_2^2S_{2-1}\right)$$

$$m_2\underset{\sim}{a}_{G_2}\cdot\left(\partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_2\right)=m\left(\tfrac{1}{4}L^2\ddot{\theta}_2+\tfrac{1}{2}L^2\ddot{\theta}_1C_{2-1}+\tfrac{1}{2}L^2\dot{\theta}_1^2S_{2-1}\right)$$

$$\underset{\approx}{I}_{G_1}\cdot\underset{\sim}{\alpha}_{B_1}=\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_1\underset{\sim}{k} \qquad \underset{\approx}{I}_{G_2}\cdot\underset{\sim}{\alpha}_{B_2}=\left(\tfrac{1}{12}mL^2\right)\ddot{\theta}_2\underset{\sim}{k}$$

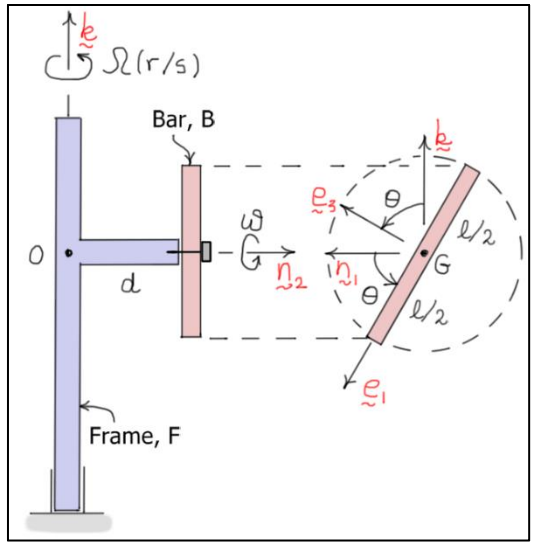
$$F_{\theta_1}=\left(-mg\,\underset{\sim}{j}\right)\cdot\left(\partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_1\right)+\left(-mg\,\underset{\sim}{j}\right)\cdot\left(\partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_1\right)=-\tfrac{3}{2}mgLS_1$$

$$F_{\theta_2}=\left(-mg\,\underset{\sim}{j}\right)\cdot\left(\partial\underset{\sim}{v}_{G_1}\big/\partial\dot{\theta}_2\right)+\left(-mg\,\underset{\sim}{j}\right)\cdot\left(\partial\underset{\sim}{v}_{G_2}\big/\partial\dot{\theta}_2\right)=-\tfrac{1}{2}mgLS_2$$

Substituting into d'Alembert's principle gives the two differential EOM

$$\left(\tfrac{4}{3}mL^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}mL^2C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}mL^2S_{2-1}\right)\dot{\theta}_2^2 + \left(\tfrac{3}{2}mgL\right)S_1 = 0$$
$$\left(\tfrac{1}{2}mL^2C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}mL^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}mL^2S_{2-1}\right)\dot{\theta}_1^2 + \left(\tfrac{1}{2}mgL\right)S_2 = 0$$

2. **Example System II** (from Intermediate Dynamics)**:** Find the ***EOM*** of the ***bar*** using d'Alembert's principle, given that the frame F is light (massless), the bar B has mass m and length ℓ, the motor torque $M_\theta(t)$ is applied between the frame and the bar, and that the motor torque $M_\phi(t)$ is applied between the ground and the frame F. Use ϕ $(\dot{\phi}=\Omega)$ and θ $(\dot{\theta}=\omega)$ as the generalized coordinates.



Reference frames: $F:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$, $B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$

d'Alembert's Principle:

$$\left(m_B \underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial\dot{\phi}}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial\dot{\phi}} = F_\phi$$
$$\left(m_B \underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial\dot{\theta}}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial\dot{\theta}} = F_\theta$$

Here,

$\underset{\sim}{\omega}_F=\dot{\phi}\underset{\sim}{k}$ $\quad$ $\partial\underset{\sim}{\omega}_F/\partial\dot{\phi}=\underset{\sim}{k}$ $\quad$ $\partial\underset{\sim}{\omega}_F/\partial\dot{\theta}=\underset{\sim}{0}$

$\underset{\sim}{\omega}_B=\dot{\theta}\underset{\sim}{n}_2+\dot{\phi}\underset{\sim}{k}$ $\quad$ $\partial\underset{\sim}{\omega}_B/\partial\dot{\phi}=\underset{\sim}{k}$ $\quad$ $\partial\underset{\sim}{\omega}_B/\partial\dot{\theta}=\underset{\sim}{n}_2=\underset{\sim}{e}_2$

$\underset{\sim}{v}_G=-d\dot{\phi}\underset{\sim}{n}_1$ $\quad$ $\partial\underset{\sim}{v}_G/\partial\dot{\phi}=-d\,\underset{\sim}{n}_1$ $\quad$ $\partial\underset{\sim}{v}_G/\partial\dot{\theta}=\underset{\sim}{0}$

$$\underset{\sim}{a}_G=-d\ddot{\phi}\underset{\sim}{n}_1-d\dot{\phi}^2\underset{\sim}{n}_2$$

$$\underset{\sim}{\alpha}_F=\ddot{\phi}\underset{\sim}{k}$$

$$\underset{\sim}{\alpha}_B=-\dot{\theta}\dot{\phi}\underset{\sim}{n}_1+\ddot{\theta}\underset{\sim}{n}_2+\ddot{\phi}\underset{\sim}{k}=-(\ddot{\phi}S_\theta+\dot{\theta}\dot{\phi}C_\theta)\underset{\sim}{e}_1+\ddot{\theta}\underset{\sim}{e}_2+(\ddot{\phi}C_\theta-\dot{\theta}\dot{\phi}S_\theta)\underset{\sim}{e}_3$$

$m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial\dot{\phi})=md^2\ddot{\phi}$ $\quad$ $m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial\dot{\theta})=0$

$\underset{\sim}{H}_G=\tfrac{1}{12}m\ell^2(\dot{\theta}\underset{\sim}{e}_2+\dot{\phi}C_\theta\underset{\sim}{e}_3)$ $\quad$ $\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B=\tfrac{1}{12}m\ell^2\left(\ddot{\theta}\underset{\sim}{e}_2+(\ddot{\phi}C_\theta-\dot{\theta}\dot{\phi}S_\theta)\underset{\sim}{e}_3\right)$

$$\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G=\tfrac{1}{12}m\ell^2\left((\dot{\phi}^2S_\theta C_\theta)\underset{\sim}{e}_2+(-\dot{\theta}\dot{\phi}S_\theta)\underset{\sim}{e}_3\right)$$

$$\left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\phi}\right) = \tfrac{1}{12}m\ell^2(\ddot{\phi}C_\theta - \dot{\theta}\dot{\phi}S_\theta)C_\theta$$

$$\left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\theta}\right) = \tfrac{1}{12}m\ell^2\ddot{\theta}$$

$$\left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\phi}\right) = -\tfrac{1}{12}m\ell^2\dot{\theta}\dot{\phi}S_\theta C_\theta$$

$$\left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\theta}\right) = \tfrac{1}{12}m\ell^2\dot{\phi}^2 S_\theta C_\theta$$

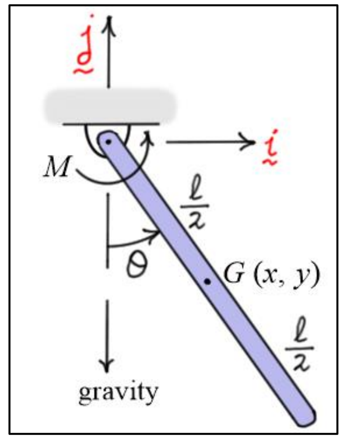
$$F_\phi = \left(M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\phi}\right) + \left(-M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\dot{\phi}\right) + \left(M_\phi \underset{\sim}{k}\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\dot{\phi}\right) = M_\phi$$

$$F_\theta = \left(M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\dot{\theta}\right) + \left(-M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\dot{\theta}\right) + \left(M_\phi \underset{\sim}{k}\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\dot{\theta}\right) = M_\theta$$

Substituting into d'Alembert's principle give the two differential EOM

$$\boxed{\begin{aligned}\left(md^2 + \tfrac{1}{12}m\ell^2C_\theta^2\right)\ddot{\phi} - \left(\tfrac{1}{6}m\ell^2S_\theta C_\theta\right)\dot{\theta}\dot{\phi} &= M_\phi \\ \left(\tfrac{1}{12}m\ell^2\right)\ddot{\theta} + \left(\tfrac{1}{12}m\ell^2S_\theta C_\theta\right)\dot{\phi}^2 &= M_\theta\end{aligned}}$$

3. **Single Link Pendulum with Constraints:** Find the EOM of the pendulum using a dependent set of generalized coordinates and d'Alembert's principle with Lagrange multipliers. The bar has mass *m* and length ℓ and is driven by a motor torque *M*. The dependent set of generalized coordinates is $(q_1, q_2, q_3) = (x, y, \theta)$.



d'Alembert's Principle:

$$\boxed{m\underset{\sim}{a}_G \cdot \frac{\partial \underset{\sim}{v}_G}{\partial \dot{q}_k} + \left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}\right)\cdot\frac{\partial \underset{\sim}{\omega}}{\partial \dot{q}_k} = F_{q_k} + \sum_{j=1}^{2}\lambda_j a_{jk}} \qquad (k = 1,\ldots,2)$$

Here,

$$\underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}/\partial\dot{\theta} = \underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}/\partial\dot{x} = \partial\underset{\sim}{\omega}/\partial\dot{y} = \underset{\sim}{0} \qquad \underset{\sim}{\alpha} = \ddot{\theta}\underset{\sim}{k}$$

$$\underset{\sim}{v}_G = \dot{x}\,\underset{\sim}{i} + \dot{y}\,\underset{\sim}{j} \qquad \partial\underset{\sim}{v}_G/\partial\dot{x} = \underset{\sim}{i} \qquad \partial\underset{\sim}{v}_G/\partial\dot{y} = \underset{\sim}{j} \qquad \partial\underset{\sim}{v}_G/\partial\dot{\theta} = \underset{\sim}{0}$$

$$\underset{\sim}{a}_G = \ddot{x}\,\underset{\sim}{i} + \ddot{y}\,\underset{\sim}{j} \qquad m\underset{\sim}{a}_G\cdot\partial\underset{\sim}{v}_G/\partial\dot{x} = m\ddot{x} \qquad m\underset{\sim}{a}_G\cdot\partial\underset{\sim}{v}_G/\partial\dot{y} = m\ddot{y} \qquad m\underset{\sim}{a}_G\cdot\partial\underset{\sim}{v}_G/\partial\dot{\theta} = 0$$

$$\left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}\right)\cdot\left(\partial\underset{\sim}{\omega}/\partial\dot{\theta}\right) = \left(\tfrac{1}{12}m\ell^2\right)\ddot{\theta}$$

$$F_\theta = -mg\,\underset{\sim}{j}\cdot\left(\partial\underset{\sim}{v}_G/\partial\dot{\theta}\right) + M\underset{\sim}{k}\cdot\left(\partial\underset{\sim}{\omega}/\partial\dot{\theta}\right) = M$$

$$F_x = -mg\,\underset{\sim}{j}\cdot\left(\partial\underset{\sim}{v}_G/\partial\dot{x}\right) + M\,\underset{\sim}{k}\cdot\left(\partial\underset{\sim}{\omega}/\partial\dot{x}\right) = 0$$

$$F_y = -mg\,\underset{\sim}{j}\cdot\left(\partial\underset{\sim}{v}_G/\partial\dot{y}\right) + M\,\underset{\sim}{k}\cdot\left(\partial\underset{\sim}{\omega}/\partial\dot{y}\right) = -mg$$

Constraints:

$$x-\tfrac{\ell}{2}S_\theta=0 \quad \text{or} \quad \dot{x}-\left(\tfrac{\ell}{2}C_\theta\right)\dot{\theta}=0 \qquad a_{11}=1,\ \ a_{12}=0,\ \ a_{13}=-\tfrac{\ell}{2}C_\theta$$

$$y+\tfrac{\ell}{2}C_\theta=0 \quad \text{or} \quad \dot{y}-\left(\tfrac{\ell}{2}S_\theta\right)\dot{\theta}=0 \qquad a_{21}=0,\ \ a_{22}=1,\ \ a_{23}=-\tfrac{\ell}{2}S_\theta$$

Substituting into d'Alembert's principle give the three differential/algebraic EOM

$$\boxed{\begin{aligned} m\ddot{x}&=\lambda_1\\ m\ddot{y}&=-mg+\lambda_2\\ \left(\tfrac{1}{12}m\ell^2\right)\ddot{\theta}&=M-\left(\tfrac{\ell}{2}C_\theta\right)\lambda_1-\left(\tfrac{\ell}{2}S_\theta\right)\lambda_2 \end{aligned}}$$

The two differentiated constraint equations are

$$\boxed{\begin{aligned} \ddot{x}-\left(\tfrac{\ell}{2}C_\theta\right)\ddot{\theta}+\left(\tfrac{\ell}{2}S_\theta\right)\dot{\theta}^2&=0\\ \ddot{y}-\left(\tfrac{\ell}{2}S_\theta\right)\ddot{\theta}-\left(\tfrac{\ell}{2}C_\theta\right)\dot{\theta}^2&=0 \end{aligned}}$$

Together, these five equations can be solved for x, y, θ, λ_1, and λ_2.

This example shows how to use Lagrange multipliers on a simple system. In this case, the equations of motion are made more complicated than if a single generalized coordinate was used. However, the use of Lagrange multipliers can simplify the equations of motion of more complex systems.