Multibody Dynamics
Generalized Speeds, Partial Angular Velocities, and Partial Velocities

<u>Generalized Speeds</u>

- To define the ***configuration*** of a multibody system with "*n*" degrees of freedom, we require at least "*n*" ***generalized coordinates***, say $q_s \; (s=1,\ldots,n)$. ***Generalized speeds*** for the system can be defined as $\dot{q}_s \; (s=1,\ldots,n)$, the time derivatives of the generalized coordinates, or as linear combinations of the time derivatives.
- Defining the "*n*" generalized speeds as linear combinations of the $\dot{q}_s \; (s=1,\ldots,n)$, write

$$u_r = \sum_{s=1}^{n}\left(Y_{rs}\dot{q}_s\right) + z_r \quad (r=1,\ldots n) \tag{1}$$

Or, in matrix notation

$$\{u\} = [Y]\{\dot{q}\} + \{z\} \tag{2}$$

- Here, the elements of the matrix $[Y]$ and the vector $\{z\}$ can be ***functions*** of the ***generalized coordinates***, $q_s \; (s=1,\ldots,n)$ ***and time***, t.
- In order for $q_s \; (s=1,\ldots,n)$ and $u_r \; (r=1,\ldots,n)$ to ***completely*** describe the configuration of the system, the matrix $[Y]$ must be ***non-singular***. Under these circumstances, Eq. (2) can be solved for $\{\dot{q}\}$ in terms of $\{u\}$ to get

$$\{\dot{q}\} = [Y]^{-1}\left(\{u\} - \{z\}\right) = [W]\{u\} + \{x\} \tag{3}$$

- Here, $[W] = [Y]^{-1}$ and $\{x\} = [Y]^{-1}\{z\}$. Eq. (3) represents a set of "*n*" ***first-order***, ***kinematical differential equations***.
- ***One example*** of generalized speeds are the ***angular velocity components*** of a rigid body. In earlier notes, the body-fixed components of the angular velocity of a rigid body were written in terms of 1-2-3 body-fixed sequence of rotation angles as

$$\{\omega'\} = \begin{bmatrix} C_2C_3 & S_3 & 0 \\ -C_2S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix}\{\dot{\theta}\} = [Y]\{\dot{\theta}\} + \{z\} \tag{4}$$

where $\left[Y\right]$ is the coefficient matrix of $\{\dot{\theta}\}$, and $\{z\}=\{0\}$.

- It was also found that the ***body-fixed components*** of the ***angular velocity*** of a rigid body can be written in terms of a set of four ***Euler parameters*** as

$$\left\{\omega'\right\}=2\left[E'\right]\left\{\dot{\varepsilon}\right\} \tag{5}$$

- Eqs. (4) and (5) can both be inverted to solve for the time derivatives of the generalized coordinates to give

$$\left\{\dot{\theta}\right\}=\begin{bmatrix} \left(C_3/C_2\right) & \left(-S_3/C_2\right) & 0 \\ S_3 & C_3 & 0 \\ \left(-C_3S_2/C_2\right) & \left(S_2S_3/C_2\right) & 1 \end{bmatrix}\left\{\omega'\right\}=\left[W\right]\left\{\omega'\right\}+\left\{x\right\} \tag{6}$$

$$\left\{\dot{\varepsilon}\right\}=\tfrac{1}{2}\left[E'\right]^T\left\{\omega'\right\}=\left[W\right]\left\{\omega'\right\}+\left\{x\right\} \tag{7}$$

- In each case, $[W]$ is the ***coefficient matrix*** of $\left\{\omega'\right\}$, and $\left\{x\right\}=\left\{0\right\}$. Recall that the coefficient matrix of Eq. (6) is ***singular*** when the second orientation angle is $\pi/2$, otherwise it is ***non-singular***. The coefficient matrix in Eq. (7) is ***non-singular*** for all body positions.

<u>Partial Angular Velocities and Partial Velocities</u>

- If $q_r\ (r=1,\ldots,n)$ are ***independent generalized coordinates*** for a ***holonomic*** system with "*n*" degrees of freedom, the ***angular velocities*** of the bodies and the ***velocities*** of the mass centers of the bodies can be written in the form

$$\underset{\sim}{\omega}_{B_k}=\sum_{r=1}^{n}\left(\underset{\sim}{\omega}_{B_k,\dot{q}_r}\,\dot{q}_r\right)+\left(\underset{\sim}{\omega}_{B_k}\right)_t \qquad \text{or} \qquad \left\{\omega_{B_k}\right\}=\left[\omega_{B_k,\dot{q}}\right]\left\{\dot{q}\right\}+\left\{\omega_{B_k}\right\}_t \tag{8}$$

$$\underset{\sim}{v}_{G_k}=\sum_{r=1}^{n}\left(\underset{\sim}{v}_{G_k,\dot{q}_r}\,\dot{q}_r\right)+\left(\underset{\sim}{v}_{G_k}\right)_t \qquad \text{or} \qquad \left\{v_{G_k}\right\}=\left[v_{G_k,\dot{q}}\right]\left\{\dot{q}\right\}+\left\{v_{G_k}\right\}_t \tag{9}$$

where $\underset{\sim}{\omega}_{B_k,\dot{q}_r}$, $\underset{\sim}{v}_{G_k,\dot{q}_r}$, $\left(\underset{\sim}{\omega}_{B_k}\right)_t$, and $\left(\underset{\sim}{v}_{G_k}\right)_t$ can be functions of $q_r\ (r=1,\ldots,n)$ and ***time***.

- The vectors $\underset{\sim}{\omega}_{B_k,\dot{q}_r}$ and $\underset{\sim}{v}_{G_k,\dot{q}_r}$ are ***partial angular velocities*** and ***partial velocities*** of the system associated with the generalized coordinates $q_r\ (r=1,\ldots,n)$.

- If $u_r \ (r=1,\ldots,n)$ is an ***independent set*** of ***generalized speeds*** as defined above, then a set of ***partial angular velocities*** and ***partial velocities*** can be defined associated with these ***generalized speeds*** as well.

$$\left\{\omega_{B_k}\right\} = \left[\omega_{B_k,u}\right]\{u\} + \left\{\bar{\omega}_{B_k}\right\}_t \tag{10}$$

$$\left\{v_{G_k}\right\} = \left[v_{G_k,u}\right]\{u\} + \left\{\bar{v}_{G_k}\right\}_t \tag{11}$$

- As before, the matrices $\left[\omega_{B_k,u}\right]$, $\left[v_{G_k,u}\right]$, and the vectors $\left\{\bar{\omega}_{B_k}\right\}_t$ and $\left\{\bar{v}_{G_k}\right\}_t$ on the right side of these equations can be ***functions*** of the ***generalized coordinates*** and ***time***.
- Defining a set of generalized speeds as in Eqs. (2) and (3), the partial angular velocities of Eqs. (8) and (10) can be related as follows

$$\begin{aligned}\left\{\omega_{B_k}\right\} &= \left[\omega_{B_k,\dot{q}}\right]\left([W]\{u\} + \{x\}\right) + \left\{\omega_{B_k}\right\}_t \\ &= \left[\omega_{B_k,\dot{q}}\right][W]\{u\} + \left(\left[\omega_{B_k,\dot{q}}\right]\{x\} + \left\{\omega_{B_k}\right\}_t\right) \\ &= \left[\omega_{B_k,u}\right]\{u\} + \left\{\bar{\omega}_{B_k}\right\}_t\end{aligned}$$

Here,

$$\boxed{\left[\omega_{B_k,u}\right] = \left[\omega_{B_k,\dot{q}}\right][W]} \tag{12}$$

$$\boxed{\left\{\bar{\omega}_{B_k}\right\}_t = \left[\omega_{B_k,\dot{q}}\right]\{x\} + \left\{\omega_{B_k}\right\}_t} \tag{13}$$

A similar result is true for the ***partial velocities***.

- Note that partial velocities and partial angular velocities are usually found by ***writing Eqs.*** (8) -(11) ***directly***, and then determining the partial velocities ***by inspection***. Eqs. (12) and (13) show, however, that we can also ***convert*** one set into the other ***given the kinematic relationships*** of Eqs. (2) and (3).

Systems with Constraints

- As discussed in earlier notes, many ***configuration*** and ***motion constraints*** can be written as

$$\sum_{k=1}^{n}\left(a_{jk}\dot{q}_k\right) + a_{j0} = 0 \qquad (j=1,\ldots,m) \tag{14}$$

- Equivalently, these constraint equations could be expressed in terms of a set of generalized speeds as

$$\sum_{s=1}^{n}\left(b_{js}u_s\right)+b_{j0}=0 \qquad (j=1,\ldots,m) \tag{15}$$

- Because Eqs. (14) and (15) represent the ***same*** constraints, it can be shown that the coefficients a_{jk} and b_{js} $(k=0,\ldots,n)$ are ***related*** as follows

$$b_{js}=\sum_{k=1}^{n}a_{jk}W_{ks} \qquad (j=1,\ldots,m;\ s=1,\ldots,n) \tag{16}$$

$$b_{j0}=a_{j0}+\sum_{k=1}^{n}a_{jk}x_k \qquad (j=1,\ldots,m) \tag{17}$$

- If the constraints equations (15) are ***independent***, then the set of linear equations can be solved for "m" of the generalized speeds in terms of the remaining "$n-m$" generalized speeds. Without loss of generality, assume the first "$n-m$" generalized speeds form an independent set, then write

$$u_{n-m+r}=\sum_{s=1}^{n-m}b'_{rs}u_s+b'_{r0} \qquad (r=1,\ldots,m) \tag{18}$$

- Given this result, all but the first "$n-m$" generalized speeds can be ***eliminated*** from the expressions for the angular velocities and mass center velocities. Then, a new ***independent*** set of partial angular velocities and partial velocities can be defined associated with the "$n-m$" independent generalized speeds.
- **Note**: ***Eliminating dependent generalized speeds*** from the angular velocities and the mass center velocities is ***not as complicated*** as ***eliminating dependent generalized coordinates***. In fact, while the resulting expressions contain only the independent generalized speeds, ***they may still contain*** the ***complete*** (dependent) ***set*** of generalized coordinates.