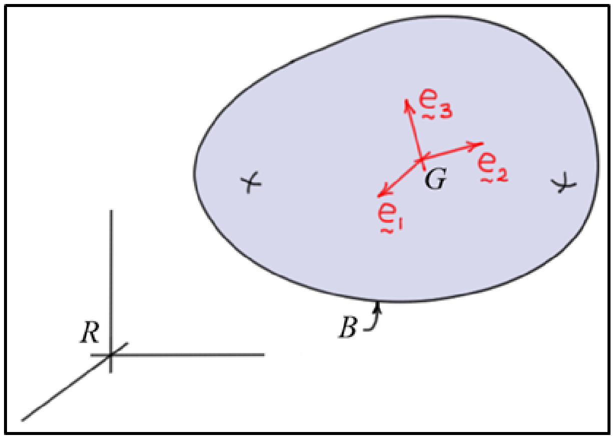
Multibody Dynamics
Examples using Kane's Equations

Examples

1. **Unconstrained Motion of a Rigid Body:** Find the equations of motion of a rigid body using Kane's equations. Use Cartesian coordinates to define the position of the mass center G and ***Euler parameters*** to define the orientation of the body. So, the vector of seven generalized coordinates is defined as follows.



$$\left[q_1, q_2, q_3, q_4, q_5, q_6, q_7\right] \triangleq \left[x_G, y_G, z_G, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\right]$$

Use the following independent set of generalized speeds.

$$\left[u_1, u_2, u_3, u_4, u_5, u_6\right] \triangleq \left[v_1', v_2', v_3', \omega_1', \omega_2', \omega_3'\right] \qquad (1)$$

Here, $v_i' \; (i=1,2,3)$ represent the ***body-fixed***, mass-center velocity components, and $\omega_i' \; (i=1,2,3)$ represent the ***body-fixed*** angular velocity components.

Solution:

Letting the $\underset{\sim}{e}_i \;\; (i=1,2,3)$ represent the ***principal directions*** for the mass-center G, and using ***body-fixed*** components of ${}^R\underset{\sim}{v}_G$ and ${}^R\underset{\sim}{\omega}_B$, write

$$\boxed{{}^R\underset{\sim}{v}_G = \sum_{i=1}^{3} v_i' \underset{\sim}{e}_i} \qquad \boxed{{}^R\underset{\sim}{\omega}_B = \sum_{i=1}^{3} \omega_i' \underset{\sim}{e}_i}$$

Using these equations, the partial velocity and partial angular velocity vectors can be written as follows for $i=1,2,3$.

$$\partial\, {}^R\underset{\sim}{v}_G \big/ \partial v_i' = \underset{\sim}{e}_i \qquad \partial\, {}^R\underset{\sim}{v}_G \big/ \partial \omega_i' = 0 \qquad \partial\, {}^R\underset{\sim}{\omega}_B \big/ \partial \omega_i' = \underset{\sim}{e}_i \qquad \partial\, {}^R\underset{\sim}{\omega}_B \big/ \partial v_i' = 0$$

The acceleration of G can be found by differentiating ${}^R\underset{\sim}{v}_G$ using the derivative rule.

$$\underset{\sim}{a}_G \triangleq \sum_{i=1}^{3} a_i' \underset{\sim}{e}_i = \frac{{}^B d}{dt}\left(\underset{\sim}{v}_G\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{v}_G\right)$$

$$= \left(\dot{v}_1' + \omega_2' v_3' - \omega_3' v_2'\right)\underset{\sim}{e}_1 + \left(\dot{v}_2' + \omega_3' v_1' - \omega_1' v_3'\right)\underset{\sim}{e}_2 + \left(\dot{v}_3' + \omega_1' v_2' - \omega_2' v_1'\right)\underset{\sim}{e}_3$$

Terms in the Equations of Motion:

$$m\,{}^R\underset{\sim}{a}_G \cdot \left(\partial\,{}^R\underset{\sim}{v}_G \big/ \partial v_i'\right) = m\,a_i' \qquad \left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right)\cdot \partial\,{}^R\underset{\sim}{\omega}_B \big/ \partial\omega_i' = \left(\sum_{j=1}^{3} I_j \dot{\omega}_j' \underset{\sim}{e}_j\right)\cdot \underset{\sim}{e}_i = I_i\dot{\omega}_i'$$

$$\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right)\cdot \partial\,{}^R\underset{\sim}{\omega}_B \big/ \partial\omega_i' = \left((I_3 - I_2)\omega_2'\omega_3'\underset{\sim}{e}_1 + (I_1 - I_3)\omega_1'\omega_3'\underset{\sim}{e}_2 + (I_2 - I_1)\omega_1'\omega_2'\underset{\sim}{e}_3\right)\cdot \underset{\sim}{e}_i$$

$$= \begin{cases} (I_3 - I_2)\omega_2'\omega_3' & (i=1) \\ (I_1 - I_3)\omega_1'\omega_3' & (i=2) \\ (I_2 - I_1)\omega_1'\omega_2' & (i=3) \end{cases}$$

Equations of Motion:

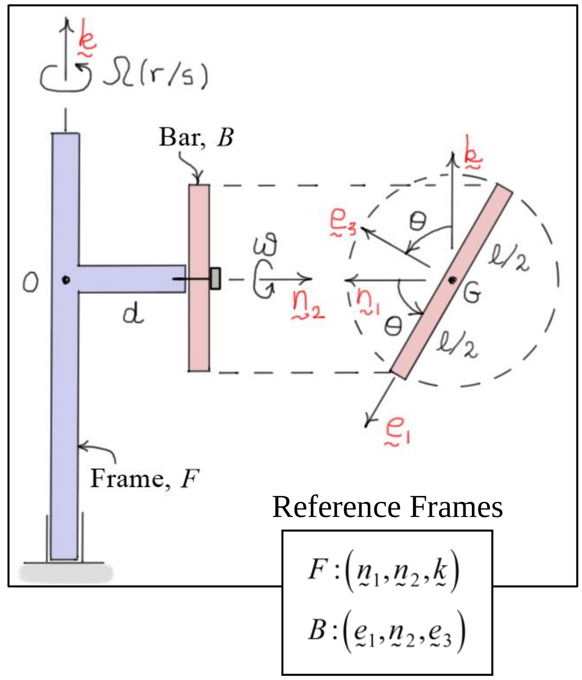
$$\boxed{\begin{aligned} m(\dot{v}_1' + \omega_2'v_3' - \omega_3'v_2') &= F_{v_1'} \\ m(\dot{v}_2' + \omega_3'v_1' - \omega_1'v_3') &= F_{v_2'} \\ m(\dot{v}_3' + \omega_1'v_2' - \omega_2'v_1') &= F_{v_3'} \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} I_1\dot{\omega}_1' + (I_3 - I_2)\omega_2'\omega_3' &= F_{\omega_1} \\ I_2\dot{\omega}_2' + (I_1 - I_3)\omega_1'\omega_3' &= F_{\omega_2} \\ I_3\dot{\omega}_3' + (I_2 - I_1)\omega_1'\omega_2' &= F_{\omega_3} \end{aligned}} \qquad (2)$$

Eqs. (2) are now ***supplemented*** with the ***kinematic differential equations***.

$$\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}\begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{z}_G \end{Bmatrix} = [R]^T \begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix} \qquad (3)$$

Together, Eqs. (2)-(3) represent a ***set*** of ***thirteen first-order***, ***ordinary differential equations*** for the ***four*** Euler parameters, ***three*** mass-center position coordinates, and the ***six*** generalized speeds defined by Eq. (1). Note the matrix $[R]^T$ converts vector components from the body frame into the base frame *R*.

2. **Example System II** (from Intermediate Dynamics): Find the equations of motion of the bar using Kane's equations, given that the frame F is light (massless), the bar B has mass m and length ℓ, the motor torque $M_\theta(t)$ is applied between the frame and the bar, and that the motor torque $M_\phi(t)$ is applied between the ground and the frame. Use $(u_1,u_2)=(v_G,\omega_2')$, where $v_G=-\underset{\sim}{n}_1\cdot\underset{\sim}{v}_G$ and $\omega_2'=\dot{\theta}=\underset{\sim}{\omega}_B\cdot\underset{\sim}{n}_2$ as the two ***independent generalized speeds***.



Solution: Using Kane's Equations

$$\boxed{\begin{aligned}\left(m_B\,\underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial v_G}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial v_G}&=F_{v_G}\\ \left(m_B\,\underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial \omega_2'}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial \omega_2'}&=F_{\omega_2'}\end{aligned}} \qquad (4)$$

Here,

$\underset{\sim}{\omega}_F=(v_G/d)\underset{\sim}{k}$ $\quad\partial\underset{\sim}{\omega}_F/\partial v_G=\underset{\sim}{k}/d$ $\quad\partial\underset{\sim}{\omega}_F/\partial\omega_2'=\underset{\sim}{0}$

$\underset{\sim}{\omega}_B=\omega_2'\,\underset{\sim}{n}_2+(v_G/d)\underset{\sim}{k}$ $\quad\partial\underset{\sim}{\omega}_B/\partial v_G=\underset{\sim}{k}/d$ $\quad\partial\underset{\sim}{\omega}_B/\partial\omega_2'=\underset{\sim}{n}_2=\underset{\sim}{e}_2$

$\underset{\sim}{v}_G=-v_G\,\underset{\sim}{n}_1$ $\quad\partial\underset{\sim}{v}_G/\partial v_G=-\underset{\sim}{n}_1$ $\quad\partial\underset{\sim}{v}_G/\partial\omega_2'=\underset{\sim}{0}$

$\underset{\sim}{a}_G=-\dot{v}_G\,\underset{\sim}{n}_1-(v_G^2/d)\underset{\sim}{n}_2$ (normal and tangential components)

$\underset{\sim}{\alpha}_F=(\dot{v}_G/d)\underset{\sim}{k}$

$\underset{\sim}{\alpha}_B=-\frac{1}{d}(\dot{v}_G S_\theta+v_G\omega_2' C_\theta)\underset{\sim}{e}_1+\dot{\omega}_2'\,\underset{\sim}{e}_2+\frac{1}{d}(\dot{v}_G C_\theta-v_G\omega_2' S_\theta)\underset{\sim}{e}_3$

$m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial v_G)=m\dot{v}_G$ $\qquad m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial\omega_2')=0$

$\underset{\sim}{H}_G=\frac{1}{12}m\ell^2(\omega_2'\,\underset{\sim}{e}_2+(v_G C_\theta/d)\underset{\sim}{e}_3)$

$\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B=\frac{1}{12}m\ell^2\left(\dot{\omega}_2'\,\underset{\sim}{e}_2+\frac{1}{d}(\dot{v}_G C_\theta-v_G\omega_2' S_\theta)\underset{\sim}{e}_3\right)$

$\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G=\frac{1}{12}m\ell^2\left((v_G^2 S_\theta C_\theta/d^2)\underset{\sim}{e}_2-(v_G\omega_2' S_\theta/d)\underset{\sim}{e}_3\right)$

$$\left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}_B\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial v_G\right) = \frac{m\ell^2}{12d^2}(\dot{v}_G C_\theta - v_G\, \omega_2'\, S_\theta) C_\theta$$

$$\left(\underset{\approx}{I}_G \cdot \underset{\sim}{\alpha}_B\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial \omega_2'\right) = \tfrac{1}{12} m \ell^2 \dot{\omega}_2'$$

$$\left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial v_G\right) = -\frac{m\ell^2}{12d^2} v_G\, \omega_2'\, S_\theta C_\theta$$

$$\left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial \omega_2'\right) = \frac{m\ell^2}{12d^2} v_G^2\, S_\theta\, C_\theta$$

$$F_{v_G} = \left(M_\theta \underset{\sim}{n}_2\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial v_G\right) + \left(-M_\theta \underset{\sim}{n}_2\right) \cdot \left(\partial \underset{\sim}{\omega}_F / \partial v_G\right) + \left(M_\phi \underset{\sim}{k}\right) \cdot \left(\partial \underset{\sim}{\omega}_F / \partial v_G\right) = M_\phi / d$$

$$F_{\omega_2'} = \left(M_\theta \underset{\sim}{n}_2\right) \cdot \left(\partial \underset{\sim}{\omega}_B / \partial \omega_2'\right) + \left(-M_\theta \underset{\sim}{n}_2\right) \cdot \left(\partial \underset{\sim}{\omega}_F / \partial \omega_2'\right) + \left(M_\phi \underset{\sim}{k}\right) \cdot \left(\partial \underset{\sim}{\omega}_F / \partial \omega_2'\right) = M_\theta$$

Substituting into Kane's equations gives the two differential equations of motion gives

$$\boxed{\begin{aligned} \left(md + \tfrac{mL^2}{12d} C_\theta^2\right)\dot{v}_G - \left(\tfrac{mL^2}{6d} S_\theta C_\theta\right) v_G \omega_2' &= M_\phi \\ \left(\tfrac{mL^2}{12}\right)\dot{\omega}_2' + \left(\tfrac{mL^2}{12d^2} S_\theta C_\theta\right) v_G^2 &= M_\theta \end{aligned}} \qquad (5)$$

Eqs. (5) are now ***supplemented*** with the ***kinematic differential equations***.

$$\boxed{\dot{\theta} = \omega_2'} \quad \text{and} \quad \boxed{\dot{\phi} = v_G / d} \qquad (6)$$

Eqs. (5) and (6) represent four, first-order ordinary differential equations in the variables θ, ϕ, v_G, and ω_2'.