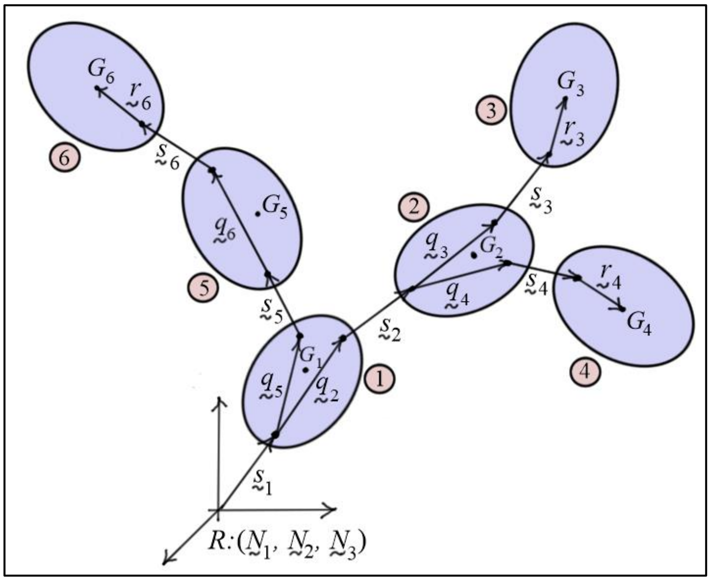
Multibody Dynamics
Equations of Motion for an Unconstrained Multibody – Example



The results below all apply to the ***six-body*** system shown. The ***orientations*** of the bodies relative to the ***inertial system*** are described using ***Euler parameters***, and the ***translation*** of the bodies ***relative*** to their ***lower-numbered body*** are defined by components of the vectors $\underset{\sim}{s}_K$ in the reference frame $\mathcal{L}(K)$. The ***angular velocities*** of the bodies are all measured ***relative*** to the ***inertial frame*** and resolved in the body frames.

In total, there are 42 first-order, kinematical differential equations and 36 first-order, dynamical differential equations.

Body Connection Array

Using the numbering of the bodies shown on the diagram, the body connection array (or lower-numbered body array) can be written as

$$\mathcal{L}(K) = (0, 1, 2, 2, 1, 5)$$

Recall that ***zero*** represents the inertial frame. The array $\left[u_K\right]$ is

$$\left(u_K\right) = \left(0, 1, 2, 2, 1, 2\right)$$

Recall that for each body, u_K is that integer such that $\mathcal{L}^{u_K}(K) = 1$. In other words, it is the number of steps down from body K to reach the system's reference body (body 1).

System State Vectors

$$\{\varepsilon\}_{24\times1} = \left[\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \ldots, \varepsilon_{61}, \varepsilon_{62}, \varepsilon_{63}, \varepsilon_{64}\right]^T \qquad \{s'\}_{18\times1} = \left[s'_{11}, s'_{12}, s'_{13}, \ldots, s'_{61}, s'_{62}, s'_{63}\right]^T$$

$$\{\omega'\}_{18\times1} = \left[\omega'_{11}, \omega'_{12}, \omega'_{13}, \ldots, \omega'_{61}, \omega'_{62}, \omega'_{63}\right]^T$$

$$\{x\}_{42\times1} = \begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} \{\varepsilon\} \\ \{s'\} \end{Bmatrix} \qquad \{y\}_{36\times1} = \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} = \begin{Bmatrix} \{\omega'\} \\ \{\dot{s}'\} \end{Bmatrix}$$

System Partial Angular Velocity Matrices

$$\left[\omega'_{1,y_1}\right]_{3\times18} = \left[\underset{1}{[I]},\underset{2}{[0]},\ldots,\underset{6}{[0]}\right] \text{ and } \left[\omega'_{1,y_2}\right]_{3\times18} = [0]$$

$$\left[\omega'_{2,y_1}\right]_{3\times18} = \left[\underset{1}{[0]},\underset{2}{[I]},\underset{3}{[0]},\ldots,\underset{6}{[0]}\right] \text{ and } \left[\omega'_{2,y_2}\right]_{3\times18} = [0]$$

$$\left[\omega'_{3,y_1}\right]_{3\times18} = \left[\underset{1}{[0]},\underset{2}{[0]},\underset{3}{[I]},\underset{4}{[0]},\ldots,\underset{6}{[0]}\right] \text{ and } \left[\omega'_{3,y_2}\right]_{3\times18} = [0]$$

$$\vdots$$

Here, $[I]$ represents the 3×3 identity matrix, and $[0]$ represents a 3×3 zero matrix.

Position Vectors of the Mass Centers of the Bodies

$$\{p_1\} = \{s_1\} + [R_1]^T\{r'_1\}$$

$$\{p_2\} = \{s_1\} + [R_1]^T\{q'_2 + s'_2\} + [R_2]^T\{r'_2\}$$

$$\{p_3\} = \{s_1\} + [R_1]^T\{q'_2 + s'_2\} + [R_2]^T\{q'_3 + s'_3\} + [R_3]^T\{r'_3\}$$

$$\{p_4\} = \{s_1\} + [R_1]^T\{q'_2 + s'_2\} + [R_2]^T\{q'_4 + s'_4\} + [R_4]^T\{r'_4\}$$

$$\{p_5\} = \{s_1\} + [R_1]^T\{q'_5 + s'_5\} + [R_5]^T\{r'_5\}$$

$$\{p_6\} = \{s_1\} + [R_1]^T\{q'_5 + s'_5\} + [R_5]^T\{q'_6 + s'_6\} + [R_6]^T\{r'_6\}$$

Velocity Vectors of the Mass Centers of the Bodies

$$\{v_1\} = \{\dot{s}_1\} - [R_1]^T[\tilde{r}'_1][\omega'_{1,y}]\{y\}$$

$$\{v_2\} = \{\dot{s}_1\} + [R_1]^T\{\dot{s}'_2\} - [R_1]^T[\tilde{q}'_2 + \tilde{s}'_2][\omega'_{1,y}]\{y\} - [R_2]^T[\tilde{r}'_2][\omega'_{2,y}]\{y\}$$

$$\{v_3\} = \{\dot{s}_1\} + [R_1]^T\{\dot{s}'_2\} + [R_2]^T\{\dot{s}'_3\} -$$
$$[R_1]^T[\tilde{q}'_2 + \tilde{s}'_2][\omega'_{1,y}]\{y\} - [R_2]^T[\tilde{q}'_3 + \tilde{s}'_3][\omega'_{2,y}]\{y\} - [R_3]^T[\tilde{r}'_3][\omega'_{3,y}]\{y\}$$

$$\{v_4\} = \{\dot{s}_1\} + [R_1]^T\{\dot{s}'_2\} + [R_2]^T\{\dot{s}'_4\} -$$
$$[R_1]^T[\tilde{q}'_2 + \tilde{s}'_2][\omega'_{1,y}]\{y\} - [R_2]^T[\tilde{q}'_4 + \tilde{s}'_4][\omega'_{2,y}]\{y\} - [R_4]^T[\tilde{r}'_4][\omega'_{4,y}]\{y\}$$

$$\{v_5\} = \{\dot{s}_1\} + \left[R_1\right]^T \left\{\dot{s}_5'\right\} - \left[R_1\right]^T \left[\tilde{q}_5' + \tilde{s}_5'\right]\left[\omega_{1,y}'\right]\{y\} - \left[R_5\right]^T \left[\tilde{r}_5'\right]\left[\omega_{5,y}'\right]\{y\}$$

$$\{v_6\} = \{\dot{s}_1\} + \left[R_1\right]^T \left\{\dot{s}_5'\right\} + \left[R_5\right]^T \left\{\dot{s}_6'\right\} -$$

$$\left[R_1\right]^T \left[\tilde{q}_5' + \tilde{s}_5'\right]\left[\omega_{1,y}'\right]\{y\} - \left[R_5\right]^T \left[\tilde{q}_6' + \tilde{s}_6'\right]\left[\omega_{5,y}'\right]\{y\} - \left[R_6\right]^T \left[\tilde{r}_6'\right]\left[\omega_{6,y}'\right]\{y\}$$

System Partial Velocity Matrices

$$\left[v_{1,y_1}\right]_{3\times18} = \left[-[R_1]^T[\tilde{r}_1'],[0],[0],[0],[0],[0]\right]$$

$$\left[v_{2,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{q}_2' + \tilde{s}_2'\right], -\left[R_2\right]^T\left[\tilde{r}_2'\right],[0],[0],[0],[0]\right]$$

$$\left[v_{3,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{q}_2' + \tilde{s}_2'\right], -\left[R_2\right]^T\left[\tilde{q}_3' + \tilde{s}_3'\right], -\left[R_3\right]^T\left[\tilde{r}_3'\right],[0],[0],[0]\right]$$

$$\left[v_{4,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{q}_2' + \tilde{s}_2'\right], -\left[R_2\right]^T\left[\tilde{q}_4' + \tilde{s}_4'\right],[0], -\left[R_4\right]^T\left[\tilde{r}_4'\right],[0],[0]\right]$$

$$\left[v_{5,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{q}_5' + \tilde{s}_5'\right],[0],[0],[0], -\left[R_5\right]^T\left[\tilde{r}_5'\right],[0]\right]$$

$$\left[v_{6,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{q}_5' + \tilde{s}_5'\right],[0],[0],[0], -\left[R_5\right]^T\left[\tilde{q}_6' + \tilde{s}_6'\right], -\left[R_6\right]^T\left[\tilde{r}_6'\right]\right]$$

$$\left[v_{1,y_2}\right]_{3\times18} = \left[[I],[0],[0],[0],[0],[0]\right]$$

$$\left[v_{2,y_2}\right]_{3\times18} = \left[[I],[R_1]^T,[0],[0],[0],[0]\right]$$

$$\left[v_{3,y_2}\right]_{3\times18} = \left[[I],[R_1]^T,[R_2]^T,[0],[0],[0]\right]$$

$$\left[v_{4,y_2}\right]_{3\times18} = \left[[I],[R_1]^T,[0],[R_2]^T,[0],[0]\right]$$

$$\left[v_{5,y_2}\right]_{3\times18} = \left[[I],[0],[0],[0],[R_1]^T,[0]\right]$$

$$\left[v_{6,y_2}\right]_{3\times18} = \left[[I],[0],[0],[0],[R_1]^T,[R_5]^T\right]$$

Time Derivatives of the Partial Angular Velocity Matrices

$$\left[\dot{\omega}_{K,y_1}'\right] = [0] \quad \text{and} \quad \left[\dot{\omega}_{K,y_2}'\right] = [0] \qquad K = 1,\ldots,6$$

Time Derivatives of the Partial Velocity Matrices

$$\left[\dot{v}_{1,y_1}\right]_{3\times18} = \left[-\left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{r}_1'\right],[0],[0],[0],[0],[0]\right]$$

$$\left[\dot{v}_{2,y_1}\right]_{3\times18} = \left[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_2'\right] - \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_2' + \tilde{s}_2'\right]\right), -\left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[\tilde{r}_2'\right], [0], [0], [0], [0]\right]$$

$$\left[\dot{v}_{3,y_1}\right]_{3\times18} = \left[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_2'\right] - \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_2' + \tilde{s}_2'\right]\right), \left(-\left[R_2\right]^T\left[\dot{\tilde{s}}_3'\right] - \left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[\tilde{q}_3' + \tilde{s}_3'\right]\right), -\left[R_3\right]^T\left[\tilde{\omega}_3'\right]\left[\tilde{r}_3'\right], [0], [0], [0]\right]$$

$$\left[\dot{v}_{4,y_1}\right]_{3\times18} = \left[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_2'\right] - \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_2' + \tilde{s}_2'\right]\right), \left(-\left[R_2\right]^T\left[\dot{\tilde{s}}_4'\right] - \left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[\tilde{q}_4' + \tilde{s}_4'\right]\right), [0], -\left[R_4\right]^T\left[\tilde{\omega}_4'\right]\left[\tilde{r}_4'\right], [0], [0]\right]$$

$$\left[\dot{v}_{5,y_1}\right]_{3\times18} = \left[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_5'\right] - \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_5' + \tilde{s}_5'\right]\right), [0], [0], [0], -\left[R_5\right]^T\left[\tilde{\omega}_5'\right]\left[\tilde{r}_5'\right], [0]\right]$$

$$\left[\dot{v}_{6,y_1}\right]_{3\times18} = \left[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_5'\right] - \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_5' + \tilde{s}_5'\right]\right), [0], [0], [0], \left(-\left[R_5\right]^T\left[\dot{\tilde{s}}_6'\right] - \left[R_5\right]^T\left[\tilde{\omega}_5'\right]\left[\tilde{q}_6' + \tilde{s}_6'\right]\right), -\left[R_6\right]^T\left[\tilde{\omega}_6'\right]\left[\tilde{r}_6'\right]\right]$$

$$\left[\dot{v}_{1,y_2}\right]_{3\times18} = \left[[0], [0], [0], [0], [0], [0]\right]$$

$$\left[\dot{v}_{2,y_2}\right]_{3\times18} = \left[[0], [R_1]^T[\tilde{\omega}_1'], [0], [0], [0], [0]\right]$$

$$\left[\dot{v}_{3,y_2}\right]_{3\times18} = \left[[0], [R_1]^T[\tilde{\omega}_1'], [R_2]^T[\tilde{\omega}_2'], [0], [0], [0]\right]$$

$$\left[\dot{v}_{4,y_2}\right]_{3\times18} = \left[[0], [R_1]^T[\tilde{\omega}_1'], [0], [R_2]^T[\tilde{\omega}_2'], [0], [0]\right]$$

$$\left[\dot{v}_{5,y_2}\right]_{3\times18} = \left[[0], [0], [0], [0], [R_1]^T[\tilde{\omega}_1'], [0]\right]$$

$$\left[\dot{v}_{6,y_2}\right]_{3\times18} = \left[[0], [0], [0], [0], [R_1]^T[\tilde{\omega}_1'], [R_5]^T[\tilde{\omega}_5']\right]$$

Equations of Motion

Given the partial velocity and partial angular velocity matrices above, the dynamical equations of motion can be written as

$$\boxed{[A]\{\dot{y}\} = \{f\}} \qquad ([A] \text{ is the generalized mass matrix})$$

Here,

$$[A]=\sum_{K=1}^{6}\left(m_K\left[v_{K,y}\right]^T\left[v_{K,y}\right]+\left[\omega'_{K,y}\right]^T\left[I'_K\right]\left[\omega'_{K,y}\right]\right)$$

$$\{f\}=\sum_{K=1}^{6}\left[v_{K,y}\right]^T\left(\{F_K\}-m_K\left[\dot{v}_{K,y}\right]\{y\}\right)+\sum_{K=1}^{6}\left[\omega'_{K,y}\right]^T\left(\{M'_K\}-\left[\tilde{\omega}'_K\right]\left[I'_K\right]\{\omega'_K\}\right)$$

The above equations represent 36 dynamical differential equations that must be supplemented with 42 kinematical differential equations.