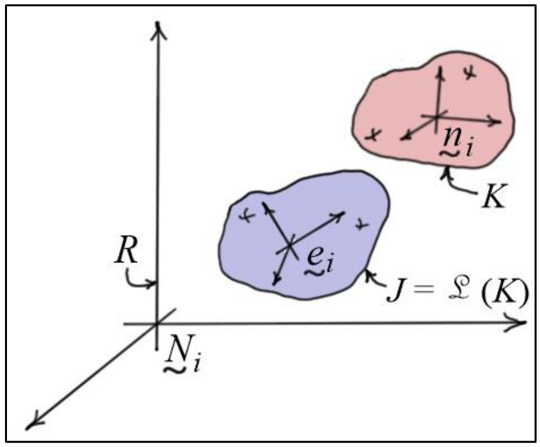
Multibody Dynamics
Time Derivative of Relative Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider two bodies of a multibody system. The unit vector set $\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$ is fixed in body K, and the set $\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$ is fixed in body J. Body J is the lower-numbered body of K ($J = \mathscr{L}(K)$). Both bodies are moving in a fixed frame $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$. If $\underset{\sim}{r}$ is a vector fixed in the body K, then, using the summation rule for angular velocities, the time derivative of $\underset{\sim}{r}$ can be written as



$$\boxed{\frac{{}^R d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}} = {}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r} = \left({}^R\underset{\sim}{\omega}_J + {}^J\underset{\sim}{\omega}_K\right) \times \underset{\sim}{r} = \left({}^R\underset{\sim}{\omega}_J \times \underset{\sim}{r}\right) + \left({}^J\underset{\sim}{\omega}_K \times \underset{\sim}{r}\right)} \tag{1}$$

When performing the cross products, the individual vectors and the resulting cross products can be expressed in any reference frame. Two cases are considered below – components of ${}^J\underset{\sim}{\omega}_K$ in body J (Case 1) and components of ${}^J\underset{\sim}{\omega}_K$ in body K (Case 2). Components of vectors in body K have been annotated with a "prime".

The transformation matrices associated with the two bodies ($[R_J]$, $[R_K]$, $\left[{}^J R_K\right]$) are defined by the following equations.

$$\boxed{\{\underset{\sim}{e}\} = [R_J]\{\underset{\sim}{N}\}} \qquad \boxed{\{\underset{\sim}{n}\} = [R_K]\{\underset{\sim}{N}\}} \qquad \boxed{\{\underset{\sim}{n}\} = \left[{}^J R_K\right]\{\underset{\sim}{e}\}}$$

Case 1:

Let $\dot{\underset{\sim}{r}}$, ${}^R\underset{\sim}{\omega}_J$, and ${}^R\underset{\sim}{\omega}_K$ be expressed in $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$, ${}^J\underset{\sim}{\omega}_K$ be expressed in $J:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$, and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$. Then,

$$\boxed{\{r\} = [R_K]^T\{r'\}} \qquad \boxed{\{\dot{r}\} = \left[\dot{R}_K\right]^T\{r'\}}$$

$$\boxed{{}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \to [\tilde{\omega}_K][R_K]^T\{r'\}} \quad \boxed{{}^R\underset{\sim}{\omega}_J \times \underset{\sim}{r} \quad \to [\tilde{\omega}_J][R_K]^T\{r'\}}$$

$${}^{J}\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \rightarrow [R_J]^T \left[{}^{J}\tilde{\omega}_K \right] \left[{}^{J}R_K \right]^T \{r'\}$$

Substituting into Eq. (1) gives

$$\left[\dot{R}_K \right]^T = [\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_J]^T \left[{}^{J}\tilde{\omega}_K \right] \left[{}^{J}R_K \right]^T$$

$$\Rightarrow \left([\tilde{\omega}_K] - [\tilde{\omega}_J] \right) [R_K]^T = [R_J]^T \left[{}^{J}\tilde{\omega}_K \right] \left[{}^{J}R_K \right]^T$$

or

$$[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T \right) = \left[{}^{J}R_K \right] \left[{}^{J}\tilde{\omega}_K \right]^T [R_J] \qquad (2)$$

<u>Case 2</u>:

Let $\dot{\underset{\sim}{r}}$, ${}^{R}\underset{\sim}{\omega}_J$, and ${}^{R}\underset{\sim}{\omega}_K$ be expressed in $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$, and let ${}^{J}\underset{\sim}{\omega}_K$ and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3 \right)$. Then,

$$\{r\} = [R_K]^T \{r'\} \qquad \{\dot{r}\} = \left[\dot{R}_K \right]^T \{r'\}$$

$${}^{R}\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \rightarrow [\tilde{\omega}_K][R_K]^T \{r'\} \qquad {}^{R}\underset{\sim}{\omega}_J \times \underset{\sim}{r} \quad \rightarrow [\tilde{\omega}_J][R_K]^T \{r'\}$$

$${}^{J}\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \rightarrow [R_K]^T \left[{}^{J}\tilde{\omega}'_K \right] \{r'\}$$

Substituting into Eq. (1) gives

$$\left[\dot{R}_K \right]^T = [\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_K]^T \left[{}^{J}\tilde{\omega}'_K \right]$$

$$\Rightarrow \left([\tilde{\omega}_K] - [\tilde{\omega}_J] \right) [R_K]^T = [R_K]^T \left[{}^{J}\tilde{\omega}'_K \right]$$

or

$$[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T \right) = \left[{}^{J}\tilde{\omega}'_K \right]^T [R_K] \qquad (3)$$

<u>Time Derivative of the Transformation Matrices Between Two Body Frames</u>

The above results can be used to determine ***two different forms*** of the ***time derivative*** of the ***relative transformation matrix*** $\left[{}^{J}R_K \right]$ depending on what reference frame is used to express the relative angular velocity vector ${}^{J}\underset{\sim}{\omega}_K$. First, note the time derivative of $\left[{}^{J}R_K \right]$ can be written as follows.

$$\left[{}^J\dot{R}_K\right] = \frac{{}^R d}{dt}\left[{}^J R_K\right] = \frac{{}^R d}{dt}\left([R_K][R_J]^T\right) = \left[\dot{R}_K\right][R_J]^T + [R_K]\left[\dot{R}_J\right]^T$$
$$= [R_K][\tilde{\omega}_K]^T[R_J]^T + [R_K][\tilde{\omega}_J][R_J]^T$$

As a skew-symmetric matrix, $[\tilde{\omega}_J] = -[\tilde{\omega}_J]^T$, so the above equation can be rewritten as

$$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T} \qquad (4)$$

Now, substituting from Eq. (2) into Eq. (4) gives

$$\left[{}^J\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T$$
$$= \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T[R_J][R_J]^T$$

$$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T}$$ (components of ${}^J\underset{\sim}{\omega}_K$ are resolved in body J)

And, substituting from Eq. (3) into Eq. (4) gives

$$\left[{}^J\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T$$
$$= \left[{}^J\tilde{\omega}'_K\right]^T[R_K][R_J]^T$$

$$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^J R_K\right]}$$ (components of ${}^J\underset{\sim}{\omega}_K$ are resolved in body K)