Intermediate Dynamics
Lagrange's Equations for Multi-Degree-of-Freedom Systems with Dependent Generalized Coordinates

Often it is more convenient to use a ***dependent set*** of ***generalized coordinates*** $q_k \ (k=1,\ldots,n)$ to define the configuration of a mechanical system. If the system possesses $N = n - m$ degrees of freedom (DOF), then there are "*m*" constraint equations that can be written in the following form.

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} \; = \; 0} \quad (j=1,\ldots,m) \qquad \text{or} \qquad \boxed{[A]_{m\times n}\{\dot{q}\}_{n\times 1} + \{a_0\}_{m\times 1} = \{0\}_{m\times 1}} \qquad (1)$$

The ***differential/algebraic*** equations of motion of the system can be derived using ***Lagrange's equations*** with ***Lagrange multipliers***.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} \; = \; F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \qquad (2)$$

Here, K is the ***kinetic energy*** of the system, F_{q_k} is the ***generalized force*** associated with the generalized coordinate q_k, λ_j is the ***Lagrange multiplier*** associated with the j^{th} constraint equation, and the $a_{jk} \ (j=1,\ldots,m; \quad k=1,\ldots,n)$ are the ***coefficients*** from the constraint equations.

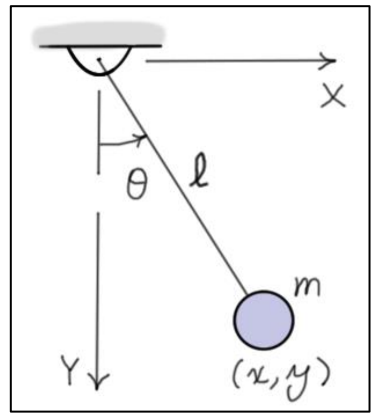
The "*n*" Lagrange's equations and the "*m*" constraint equations form a set of "$n+m$" ***differential/algebraic equations*** (DAE) for the "$n+m$" ***unknowns*** – the "*n*" ***generalized coordinates*** $q_k \ (k=1,\ldots,n)$ and the "*m*" ***Lagrange multipliers*** $\lambda_j \ (j=1,\ldots,m)$. The ***Lagrange multipliers*** are ***related*** to the ***forces*** and ***moments*** required to maintain the constraints as the system moves. It is ***important*** that *K*, F_{q_k}, and the a_{jk} be written ***only in terms*** of q_k, $\dot{q}_k$, and ***no other variables***.

If some of the forces and torques are ***conservative***, then their contributions to the equations of motion can be calculated in terms of ***potential energy functions***. In this case, the ***differential/algebraic*** EOM can be derived from the equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \; = \; \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \qquad (3)$$

Here, $L = K - V$ is the ***Lagrangian*** of the system, V is the ***potential energy function*** for the ***conservative*** forces and torques, $\left(F_{q_k}\right)_{nc}$ is the ***generalized force*** associated with q_k for the ***non-conservative*** forces and torques, only, λ_j is the ***Lagrange multiplier*** associated with the j^{th} constraint equation, and a_{jk} $(j=1,\ldots m;\, k=1,\ldots n)$ represents the ***coefficients*** from the ***constraint equations***.

As before, the "n" Lagrange's equations and the "m" constraint equations form a set of "$n+m$" ***differential/algebraic equations*** for the "$n+m$" unknowns – the "n" ***generalized coordinates*** q_k $(k=1,\ldots,n)$ and the "m" ***Lagrange multipliers*** λ_j $(j=1,\ldots,m)$.



Example: Equations of Motion of the Simple Pendulum

Approach #1: Using θ as the generalized coordinate

The ***simple pendulum*** is a single degree of freedom (SDOF) system. Using θ as the generalized coordinate, the Lagrangian of the system can be written as follows.

$$L = \tfrac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta)$$

Using Lagrange's equations ***without constraints***, the equation of motion is found to be

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0 \qquad (4)$$

Approach #2: Using (x, y) as the generalized coordinates

To use two generalized coordinates for a SDOF system, ***Lagrange's equations with a single constraint*** must be used. In this case, the configuration constraint is

$$x^2 + y^2 = \ell^2$$

Differentiating this equation, the constraint can be written in the form of Eq. (1).

$$x\dot{x} + y\dot{y} = 0 \qquad \text{or} \qquad \begin{bmatrix} x & y \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = 0 \qquad (5)$$

The ***coefficients*** a_{jk} $(j=1;\, k=0,1,2)$ are defined as follows.

$$[A] = \begin{bmatrix} x & y \end{bmatrix} \quad \Rightarrow \quad a_{11} = x \quad a_{12} = y \quad \text{and} \quad a_{10} = 0 \qquad (6)$$

Lagrange's equations ***with constraints*** can then be written in the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\lambda a_{1k}} \qquad k=1,2 \tag{7}$$

The ***Lagrangian*** can be written in terms of the generalized coordinates and their derivatives as

$$\boxed{L=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+mg\,y}$$

Using Eq. (7) and the definitions of the constraint equation coefficients in Eq. (6), the following ***two coupled, second-order, differential/algebraic equations*** (DAE) can be found. The Lagrange multiplier λ appears as an algebraic unknown in the equations.

$$\boxed{\begin{aligned} m\ddot{x}-\lambda x&=0 \\ m\ddot{y}-mg-\lambda y&=0 \end{aligned}} \tag{8}$$

Differentiating the constraint Eq. (5) again, and combining it with Eqs. (8) gives the following ***complete set*** of ***three coupled, second-order, differential/algebraic equations*** in three unknowns (x,y,λ).

$$\boxed{\begin{aligned} m\ddot{x}-\lambda x&=0 \\ m\ddot{y}-mg-\lambda y&=0 \\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2&=0 \end{aligned}} \tag{9}$$

Alternatively, the ***Lagrange multiplier*** λ can be eliminated from Eqs. (8) to form a single differential equation. For example, multiply the first equation by "*y*" and the second equation by "*x*" and subtract the two equations. Following this approach, the equations of motion are written as ***two coupled, second-order differential equations*** with no algebraic unknowns.

$$\boxed{\begin{aligned} y\ddot{x}-x\ddot{y}-gx&=0 \\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2&=0 \end{aligned}} \tag{10}$$