Intermediate Dynamics
Constraint Relaxation Method: The Meaning of Lagrange Multipliers

Previously, it was noted that if a dynamic system is described using "n" ***generalized coordinates*** $q_k\ (k=1,\ldots,n)$, and if the system is subjected to "m" ***independent configuration constraint equations*** of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} \;=\; 0} \qquad (j=1,\ldots,m) \tag{1}$$

then, the equations of motion of the system can be found by using one of the following two forms of ***Lagrange's equations*** with ***Lagrange multipliers***.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} \;=\; F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{2}$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \;=\; \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{3}$$

Here, K is the ***kinetic energy*** of the system, F_{q_k} is the ***generalized force*** associated with the generalized coordinate q_k, L is the ***Lagrangian*** of the system, V is the ***potential energy function*** for the **conservative forces** and ***torques***, $\left(F_{q_k}\right)_{nc}$ is the ***generalized force*** associated with q_k for the ***non-conservative forces*** and ***torques***, only, λ_j is the ***Lagrange multiplier*** associated with the j^{th} ***constraint equation***, and $a_{jk}\ (j=1,\ldots,m;\ k=1,\ldots,n)$ are the ***coefficients*** from the ***constraint equations***. Eqs. (1) along with Eqs. (2) or (3) form a set of "$n+m$" ***differential/algebraic equations*** for the "n" ***generalized coordinates*** and the "m" ***Lagrange multipliers***.

Alternatively, some or all the constraints can be ***relaxed*** (or ***removed***), and they can be ***replaced*** with ***force*** and/or ***torque components*** that are required to maintain the constraints. Then, the "n" Lagrange's equations can be formulated in terms of the "n" generalized coordinates and the "m" constraint force (and/or torque) components. Together with the constraint equations, this forms a ***complete set*** of "$n+m$" ***differential/algebraic equations*** for the "n" generalized coordinates and the "m" constraint force and/or torque components. If all the constraints are relaxed, then Eqs. (2) and (3) can be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right)-\frac{\partial K}{\partial q_k} = F_{q_k}+\left(F_{q_k}\right)_{\text{constraints}}} \qquad (k=1,\ldots,n) \qquad (4)$$

and

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc}+\left(F_{q_k}\right)_{\text{constraints}}} \qquad (k=1,\ldots,n) \qquad (5)$$

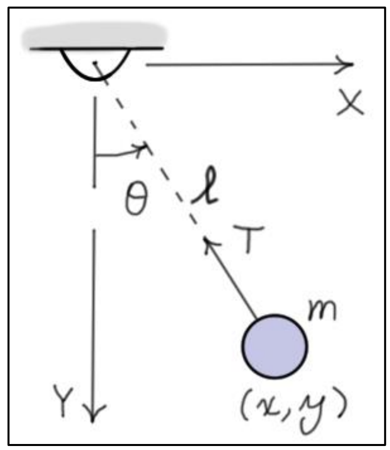
Comparing Eqs. (2) and (4) or Eqs. (3) and (5) gives the following.

$$\boxed{\left(F_{q_k}\right)_{\text{constraints}} = \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \qquad (6)$$

So, the ***Lagrange multipliers*** are ***directly related*** to the ***forces*** and/or ***torques*** required to maintain the constraints.

Example: The Simple Pendulum

To illustrate constraint relaxation, consider the simple pendulum shown with $q_1 = x$ and $q_2 = y$ as the generalized coordinates. The fixed length of the pendulum can be used to ***relate*** the two coordinates, but here the ***length constraint*** will be ***relaxed*** in the formulation. Specifically, Lagrange's equations can be written in the form



$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\left(F_{q_k}\right)_{nc}+\left(F_{q_k}\right)_{\text{constraint}}} \qquad (7)$$

Here, the Lagrangian $\boxed{L=\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mgy}$, and as there are no nonconservative forces or torques, $\boxed{\left(F_{q_k}\right)_{nc}=0}$. The contributions of the ***constraint force*** to the generalized forces can be calculated as follows.

$$\boxed{\left(F_x\right)_{\text{constraint}} = \underset{\sim}{T}\cdot\left(\partial \underset{\sim}{v}/\partial \dot{x}\right)=T\left(-\left(x/\ell\right)\underset{\sim}{i}-\left(y/\ell\right)\underset{\sim}{j}\right)\cdot\left[\partial\left(\dot{x}\underset{\sim}{i}+\dot{y}\,\underset{\sim}{j}\right)\Big/\partial\dot{x}\right]=-T\left(x/\ell\right)} \qquad (8)$$

$$\boxed{\left(F_y\right)_{\text{constraint}} = \underset{\sim}{T}\cdot\left(\partial \underset{\sim}{v}/\partial \dot{y}\right)=T\left(-\left(x/\ell\right)\underset{\sim}{i}-\left(y/\ell\right)\underset{\sim}{j}\right)\cdot\left[\partial\left(\dot{x}\underset{\sim}{i}+\dot{y}\,\underset{\sim}{j}\right)\Big/\partial\dot{y}\right]=-T\left(y/\ell\right)} \qquad (9)$$

Substituting into Lagrange's equations (Eq. (7)) and supplementing with the twice-differentiated constraint equation it can be shown that the equations of motion can be written as follows.

$$\boxed{\begin{aligned} m\ddot{x} + \left(\tfrac{x}{\ell}\right)T &= 0 \\ m\ddot{y} - mg + \left(\tfrac{y}{\ell}\right)T &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \qquad (10)$$

Using ***Lagrange multipliers***, it was shown in previous notes that the equations for the pendulum could be written as

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x &= 0 \\ m\ddot{y} - mg - \lambda y &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \qquad (11)$$

Comparing Eqs. (10) and (11), the ***Lagrange multiplier*** λ is clearly equal to $-T/\ell$ the negative of the ***force per unit length*** of the ***pendulum***.