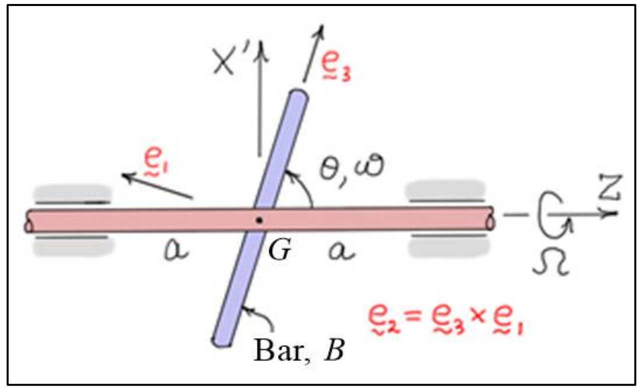
Multibody Dynamics
Exercises #9 – Kane's Equations

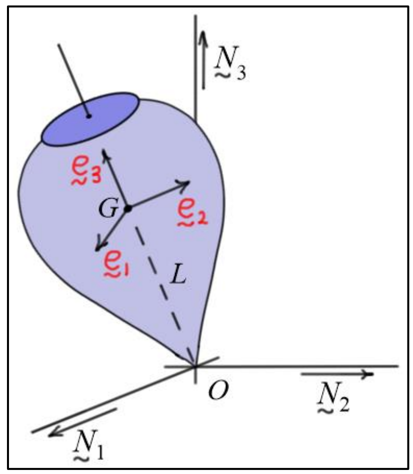
1. Using Kane's equations, find the equations of motion of the two degree-of-freedom system shown. The system consists of a slender bar B of length ℓ and mass m that is pinned through the center of a ***light*** shaft. The rotation of the shaft about the Z-axis is described by the angle $\phi\ (\dot{\phi} = \Omega)$, and the rotation of the bar B about the Y'-axis is described by the angle $\theta\ (\dot{\theta} = \omega)$.



A motor torque M_ϕ is applied to the shaft about the Z-axis, and a motor torque M_θ is applied to B about the Y'-axis. Use $(u_k) = (\omega_1', \omega_2')$ as generalized speeds, where $\omega_i' = {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{e}_i \quad (i = 1, 2)$.

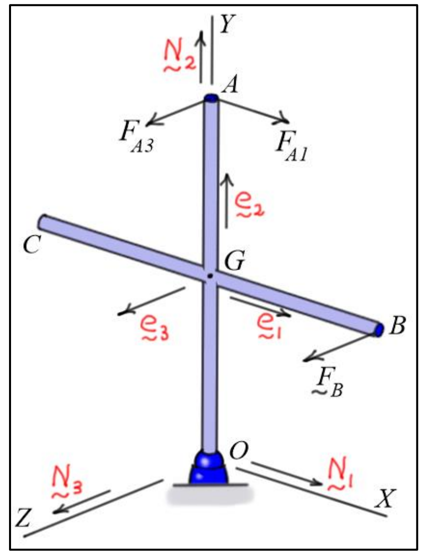
2. Spinning Top

a) Using Kane's equations, find the equations of motion of the ***three*** degree-of freedom spinning top shown in the diagram. Assume the moments of inertia of the top about the $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ directions are $I_1 = I_2 = I$, and the moment of inertia about the $\underset{\sim}{e}_3$ direction is I_3. Also, assume point O is fixed and acts like a ball-and-socket joint. Use Euler parameters to define the orientation of the top and define the generalized speeds to be $(u_k) = (\omega_1', \omega_2', \omega_3')$ the body-fixed angular velocity components, where $\omega_i' = {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{e}_i \ (i = 1, 2, 3)$. The unit vector set $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in and rotates with the top.



b) ***Begin*** to derive the equations of motion using d'Alembert's principle with Euler parameters as the generalized coordinates. Derive the equations just far enough to ***explain*** the ***complexities*** that arise resulting from the use of Euler parameters which form a ***dependent*** set of generalized coordinates. ***Note***: There is no need to complete the derivation of the equations.

3. The bracket $OABC$ shown in the diagram (shaped like a "+" sign) is attached to the ground with a ball-and-socket joint at O. The bars OA and BC are identical slender bars with mass m and length L. The orientation of the bracket is to be described using a 1-2-3 orientation angle sequence. In the configuration shown, all angles are ***zero*** so the inertial unit vectors $\left(\underset{\sim}{N}_i,\ i=1,2,3\right)$ are aligned with the body-fixed unit vectors $\left(\underset{\sim}{e}_i,\ i=1,2,3\right)$. The bracket moves under the action of its own weight at G and the external forces at A and B with $\underset{\sim}{W} = -2mg\,\underset{\sim}{N}_2$, $\underset{\sim}{F}_A = F_{A1}\,\underset{\sim}{e}_1 + F_{A3}\,\underset{\sim}{e}_3$, and $\underset{\sim}{F}_B = F_B\,\underset{\sim}{e}_3$.



The configuration of the bracket is described by the generalized coordinates $(q_k) = (\theta_1, \theta_2, \theta_3)$ and the generalized speeds $(u_k) = (\omega'_1, \omega'_2, \omega'_3)$. Here, $\theta_i\ (i=1,2,3)$ represent the orientation angles, and $\omega'_i\ (i=1,2,3)$ represent the body-fixed angular velocity components. Complete the following.

a) Identify the partial velocities of the mass center G $\left(\partial \underset{\sim}{v}_G / \partial u_k\ (k=1,2,3)\right)$, the partial velocities of A $\left(\partial \underset{\sim}{v}_A / \partial u_k\ (k=1,2,3)\right)$, the partial velocities of B $\left(\partial \underset{\sim}{v}_B / \partial u_k\ (k=1,2,3)\right)$, and the partial angular velocities of the bracket $\left(\partial \underset{\sim}{\omega} / \partial u_k\ (k=1,2,3)\right)$.

b) Find the generalized forces $F_{u_k}\ (k=1,2,3)$ associated with $(u_k) = (\omega'_1, \omega'_2, \omega'_3)$.

c) Find the three equations of motion of the bracket using Kane's equations for the set of generalized speeds $(u_k) = (\omega'_1, \omega'_2, \omega'_3)$. Express the equations in terms of the variables $(\theta_1, \theta_2, \theta_3)$, $(\omega'_1, \omega'_2, \omega'_3)$, and $(\dot{\omega}'_1, \dot{\omega}'_2, \dot{\omega}'_3)$.

d) Identify the kinematical differential equations for the 1-2-3 rotation sequence that must accompany the equations from part (c) in the solution process.

***Note*:** Whenever necessary, use the tables to expedite your work.