Introductory Control Systems
Introduction to State-Space Models

References:

F.H. Raven, *Automatic Control Engineering*, 5th Ed., McGraw-Hill, Inc., 1995.
R.C. Dorf & R.H. Bishop, *Modern Control Systems*, 11th Ed., Pearson Prentice Hall, Inc., 2008.
K. Ogata, *Modern Control Engineering*, 5th Ed., Pearson Education, Inc., 2010.

Transfer functions provide a convenient method of describing the response of single-input, single-output (SISO) systems. They represent a single differential equation with constant coefficients that relates the input and output of a linear system. To do the same for multiple-input, multiple-output (MIMO) linear systems with constant coefficients, multiple transfer functions are required, one for each input/output pair. As an alternative, state-space models can be used for SISO or MIMO systems. They can be easily modified to account for any convenient input and output signals. A state-space model is simply a set of differential equations that represent the behavior of the system expressed in state-space form.

In addition to easily accommodating multiple input and output variables, state-space models can also be used to model linear or nonlinear systems and systems with time-varying coefficients. In these notes only systems with constant coefficients are considered.

<u>State-Space Form</u>

The equations describing the dynamic behavior of a system are said to be in state-space form when they are written as

$$\dot{\underset{\sim}{x}} = [A]\underset{\sim}{x} + [B]\underset{\sim}{u} \qquad (1)$$

$$\underset{\sim}{z} = [C]\underset{\sim}{x} + [D]\underset{\sim}{u} \qquad (2)$$

Here,

$\underset{\sim}{x}(t)$ is the $N \times 1$ vector of time-varying states – the elements are called state variables

$\underset{\sim}{u}(t)$ is the $M \times 1$ vector of time-varying input signals

$\underset{\sim}{z}(t)$ is the $K \times 1$ vector of time-varying output signals

$[A]$ is an $N \times N$ constant coefficient matrix, $[B]$ is an $N \times M$ constant coefficient matrix

$[C]$ is a $K \times N$ constant coefficient matrix, $[D]$ is a $K \times M$ constant coefficient matrix

Expressing the equations in this form allows the analyst to incorporate as many input variables and output variables as needed.

Some authors include a second input vector of disturbances so unwanted input signals (disturbances) can be treated separately. In these notes, a single vector of input signals is employed.

Example:

The system shown is a double mass-spring-damper system with two degrees of freedom defined by the displacement variables (y_1, y_2) measured from the equilibrium positions of the masses. The system has two input forces represented by the variables $\left(f_1(t), f_2(t)\right)$. Using Newton's laws or Lagrange's equations, the equations of motion of the system can be written as follows.

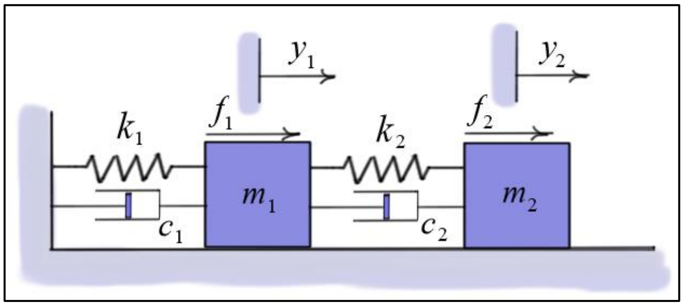


Fig. 1 Double Mass-Spring Damper System

$$\boxed{\begin{aligned} m_1\ddot{y}_1 + (c_1+c_2)\dot{y}_1 + (k_1+k_2)y_1 - c_2\,\dot{y}_2 - k_2\,y_2 &= f_1(t) \\ m_2\ddot{y}_2 - c_2\,\dot{y}_1 - k_2\,y_1 + c_2\,\dot{y}_2 + k_2\,y_2 &= f_2(t) \end{aligned}} \tag{3}$$

Eqs. (3) represent a set of coupled, second-order, linear, ordinary differential equations. Note that the system is in static equilibrium when $y_1 = y_2 = 0$.

Find: The state-space form of these differential equations of motion.

Solution:

To express Eqs. (3) in state-space form, define a set of state variables and input variables. For example,

$$x_1 \triangleq y_1 \qquad x_2 \triangleq y_2 \qquad x_3 \triangleq \dot{y}_1 \qquad x_4 \triangleq \dot{y}_2 \qquad u_1 \triangleq f_1 \qquad u_2 \triangleq f_2 \tag{4}$$

Then Eqs. (3) can be rewritten as four first-order, linear differential equations.

$$\boxed{\begin{aligned} &\dot{x}_1 = x_3 \\ &\dot{x}_2 = x_4 \\ &\dot{x}_3 = \left[-(k_1+k_2)x_1 + k_2\,x_2 - (c_1+c_2)x_3 + c_2\,x_4\right]/m_1 \;+\; u_1/m_1 \\ &\dot{x}_4 = \left[k_2\,x_1 - k_2\,x_2 + c_2\,x_3 - c_2\,x_4\right]/m_2 \;+\; u_2/m_2 \end{aligned}} \tag{5}$$

Eqs. (5) can be written in the matrix form

$$\dot{\underset{\sim}{x}} = [A]\underset{\sim}{x} + [B]\underset{\sim}{u} \tag{6}$$

Here, the coefficient matrices $[A]$ and $[B]$ are defined to be

$$[A] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -(k_1+k_2)/m_1 & k_2/m_1 & -(c_1+c_2)/m_1 & c_2/m_1 \\ k_2/m_2 & -k_2/m_2 & c_2/m_2 & -c_2/m_2 \end{array}\right] \tag{7}$$

$$[B] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1/m_1 & 0 \\ 0 & 1/m_2 \end{array}\right] \tag{8}$$

The output variables can now be chosen to match the needs of the analysis. The only requirement is that they be expressible in the form of Eq. (2). For example, to output the positions of the masses and the corresponding forces of excitation, define the output equation as follows.

$$\underset{\sim}{z} = [C]\underset{\sim}{x} + [D]\underset{\sim}{u}$$

with

$$\underset{\sim}{z} = \begin{Bmatrix} x_1 \\ x_2 \\ f_1 \\ f_2 \end{Bmatrix} \qquad [C] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [D] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underset{\sim}{u} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{9}$$

To output the positions and corresponding velocities of the masses and not the input forces, define the output equation as follows.

$$\underset{\sim}{z} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \qquad [C] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [D] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underset{\sim}{u} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{10}$$

<u>Example</u>:

Problem E2.15 from Dorf & Bishop, *Modern Control Systems*, 11th edition gives the equations associated with a closed-loop spacecraft platform positioning system as shown in Eqs. (11) below. The variable p represents the platform position, variable θ represents the angle of the motor drive system, variable v_2 represents the input voltage to the motor, variable v_1 represents a voltage equal to the difference in the desired position r and the actual position p, and K is the gain of the proportional controller.

$$\boxed{\begin{aligned} \ddot{p} + 2\dot{p} + 4p &= \theta(t) \\ \dot{\theta}(t) &= 0.6 v_2 \\ v_2 &= K v_1 \\ v_1 &= r - p \end{aligned}} \tag{11}$$

<u>Find</u>: The state-space form of these differential equations of motion.

<u>Solution</u>:

First, define a set of state and input variables. For example,

$$x_1 \triangleq p \qquad x_2 \triangleq \theta \qquad x_3 \triangleq \dot{p} \qquad u \triangleq r \tag{12}$$

Rewriting in state-space form gives

$$\boxed{\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= 0.6K(u - x_1) \\ \dot{x}_3 &= -4x_1 + x_2 - 2x_3 \end{aligned}} \quad \text{matrix form} \Rightarrow \boxed{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -0.6K & 0 & 0 \\ -4 & 1 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0.6K \\ 0 \end{Bmatrix} u} \tag{13}$$

Defining output variables as

$$z_1 \triangleq p \qquad z_2 \triangleq \dot{p} \qquad z_3 \triangleq v_2 = K\left(r - p\right) \qquad z_4 \triangleq r$$

the output equation can be written as

$$\underset{\sim}{z} = [C]\underset{\sim}{x} + [D]\underset{\sim}{u} \tag{14}$$

with

$$\underset{\sim}{z} = \begin{Bmatrix} p \\ \dot{p} \\ v_2 \\ r \end{Bmatrix} \qquad [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [D] = \begin{bmatrix} 0 \\ 0 \\ K \\ 1 \end{bmatrix} r \tag{15}$$

State Equation Diagrams

State equations such as those presented in Eqs. (5) and (13) above can be solved numerically by coding them directly into Simulink diagrams in MATLAB®. Fig. 2 below shows a Simulink diagram for Eq. (5) with sinusoidal force input signals on each mass and the positions of and forces on the masses as the four output signals. Of course, any signal in the diagram can be viewed as an output of the model. Note the symbol "mu1" represents $1/m_1$ and "mu2" represents $1/m_2$.

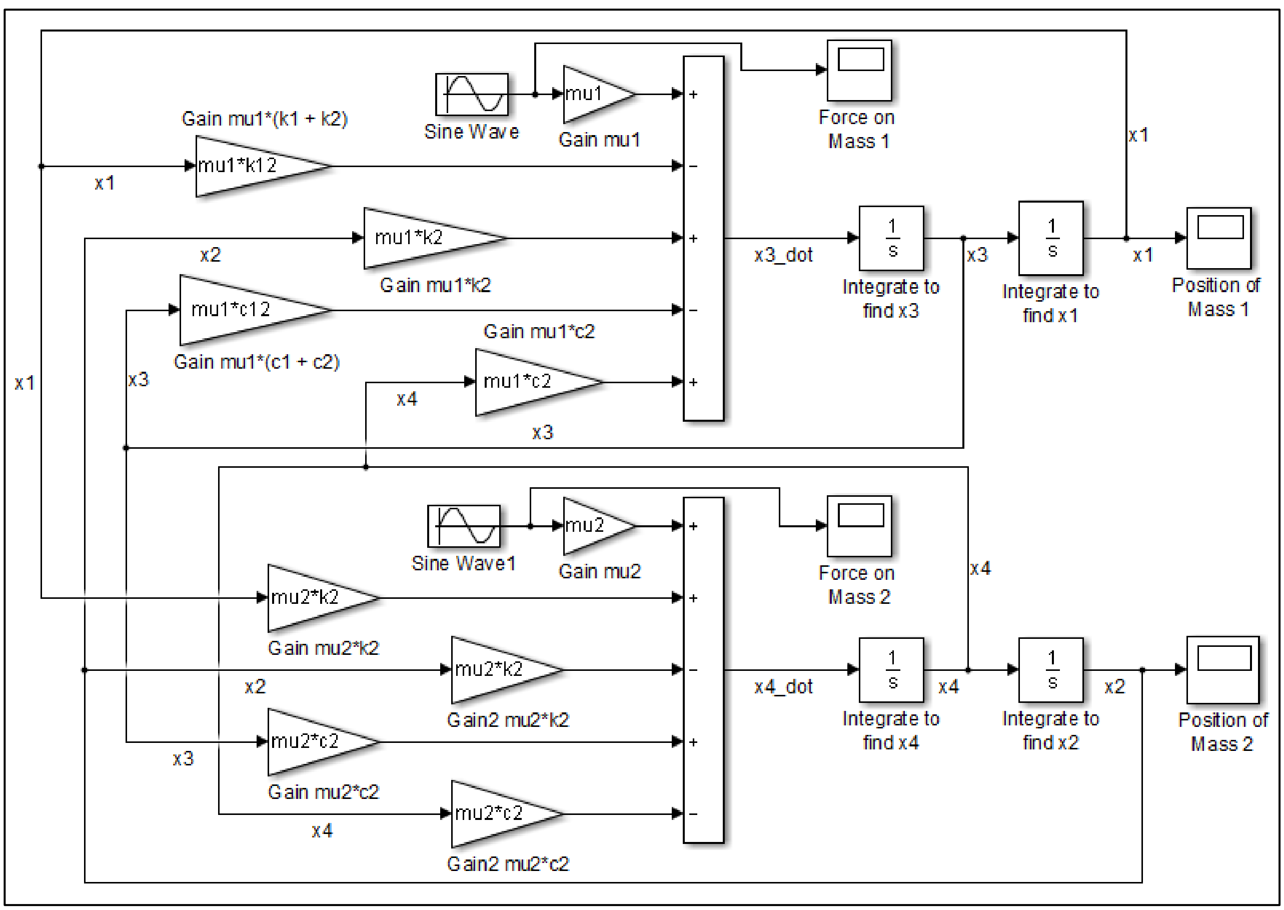


Fig. 2 Simulink Diagram for Double Mass-Spring-Damper System

The diagram can certainly be simplified using subsystems, but it is presented in its expanded form so the reader can more easily see the equivalence of the diagram and the equations presented in Eqs. (5). The two summing blocks in the center of the diagram form the sums on the right sides of the third and fourth equations. The resulting sums are the values of $\dot{x}_3$ and $\dot{x}_4$. Those signals are then integrated to find x_3 and x_4 which, in turn, are integrated to find x_1 and x_2, the positions of the two masses.

Model results are presented below for two sample cases. Case 1 is a forced response and Case 2 is a free, damped response due to nonzero initial conditions. Transfer functions can be used to generate the Case 1 forced response. However, they cannot be used to generate the Case 2 response because transfer functions are defined assuming zero initial conditions.

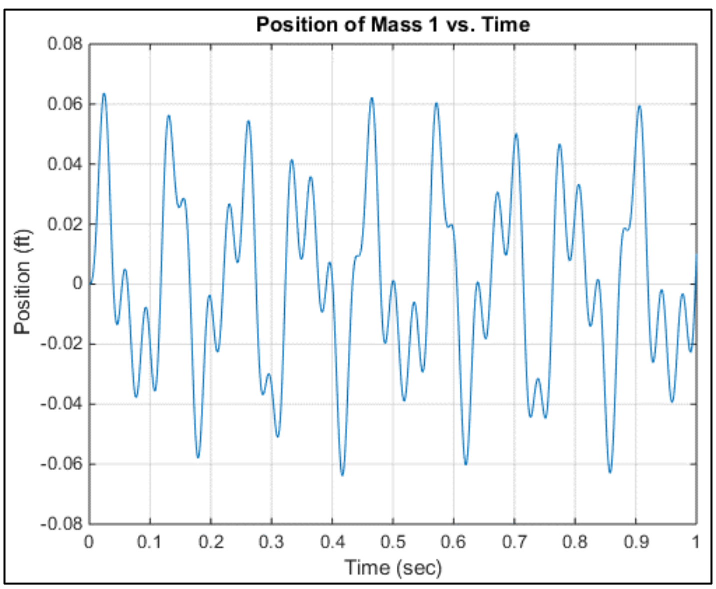
Case 1: (forced response)

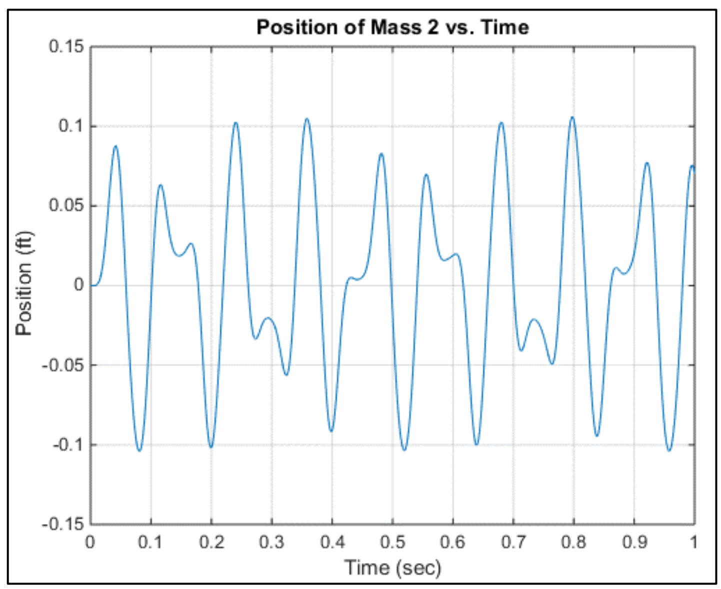
In Case 1 a sinusoidal force is applied to mass m_1. The system starts from its equilibrium position where all initial conditions are zero. The physical parameters and forcing functions used for this case are as follows.

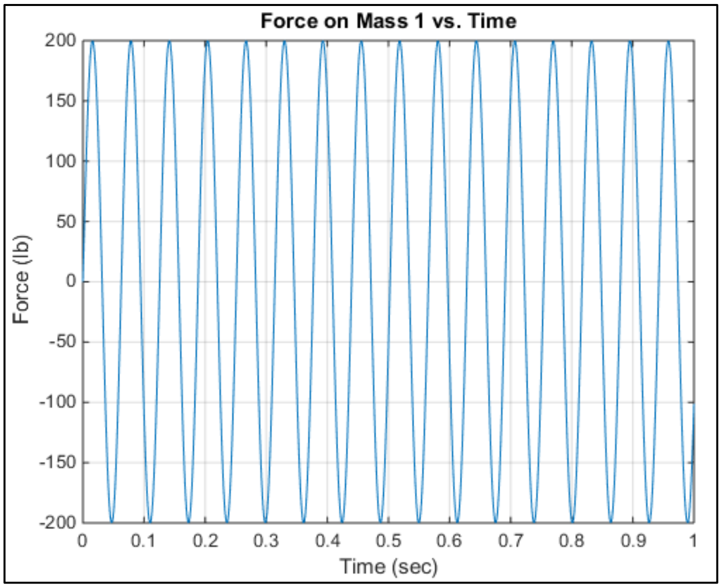
$m_1 = 0.2$ (slugs) $\quad m_2 = 0.4$ (slugs) $\quad k_1 = k_2 = 3000$ (lb/ft) $\quad c_1 = c_2 = 0$

$f_1(t) = 200\sin(100t)$ (lb) $\quad f_2(t) = 0$ (lb)

The three figures below show model results for this case for the first one-second of the response. The positions of the masses are plotted in the first two figures and the forcing function $f_1(t)$ is plotted in the third figure. Note that even though the forcing function is sinusoidal (harmonic), the responses of the two masses are not.







Case 2: (free, damped response)

In Case 2 the system is located away from its equilibrium position by applying nonzero initial conditions in the integration blocks of the Simulink model. The system is then allowed to respond freely with $f_1(t) = f_2(t) = 0$ as it returns to equilibrium. The physical parameters and initial conditions for this case are as follows.

$m_1 = 0.2$ (slugs) $m_2 = 0.4$ (slugs) $k_1 = k_2 = 3000$ (lb/ft) $c_1 = 5$ (lb-s/ft) $c_2 = 2$ (lb-s/ft)

$x_1(0) = 1$ (ft) $x_2(0) = 0.5$ (ft) $\dot{x}_1(0) = 0$ (ft/s) $\dot{x}_2(0) = 0$ (ft/s)

The two figures below show the positions of the masses for the first one-second of the response. Note that after an initial transient response (the first 0.2 seconds, or so), both masses settle into a damped harmonic response at about 9 Hertz.