Summary

This unit defines moments and products of inertia for rigid bodies and shows how they are used to form inertia matrices (or dyadics). Inertia matrices are then used to calculate principal moments of inertia and principal directions. More generally, it shows how to transform the components of inertia dyadics from one set of reference axes to another. Finally, it defines angular momentum vectors and the kinetic energy function for rigid bodies and shows how to use inertia matrices to compute them.

An Addendum is included to discuss the special case of nondistinct (equal) principal moments of inertia and their associated eigenvectors. The principal moments of inertia and the principal directions of a square prism are presented as an example.

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Moments and Products of Inertia and the Inertia Matrix

Moments of Inertia

Consider the rigid body $B$ is shown in the diagram. The unit vectors $B : (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are fixed in $B$ and are directed along a convenient set of axes $(x, y, z)$ that pass through the mass center $G$. The moments of inertia of the body about these axes are defined as follows.

\[
\begin{align*}
I_{xx}^G &= \int_B (y^2 + z^2) \, dm \\
I_{yy}^G &= \int_B (x^2 + z^2) \, dm \\
I_{zz}^G &= \int_B (x^2 + y^2) \, dm
\end{align*}
\]

Here, $x$, $y$, and $z$ are defined as the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ components of $\mathbf{r}_{P/G}$, the position vector of an arbitrary point $P$ of the body relative to $G$, that is, $\mathbf{r}_{P/G} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3$. The integrals are taken over the entire volume of the mass.

Moments of inertia of a body about an axis measure the distribution of the body’s mass about that axis and are always positive (although, if small (negligible), they can be assumed to be zero). The smaller the inertia the more concentrated the mass is about the axis. Inertia values can be found by measurement, calculation, or both. Calculations are based on direct integration, the “body build-up” technique, or both. In the body build-up technique, inertias of simple shapes are added together to estimate the inertia of a composite shape. The inertias of simple shapes (about their individual mass centers) are found in standard inertia tables. These values are transferred to axes through the composite mass center using the Parallel-Axes Theorem for Moments of Inertia.

Parallel-Axes Theorem for Moments of Inertia

The inertia $I_{ii}^A$ of a body about an axis $i$ passing through any point $A$ is equal to the sum of the inertia $I_{ii}^G$ of the body about a parallel axis through the mass center $G$ plus the mass $m$ times the square of the shortest distance $d_i$ between the two parallel axes.

\[
I_{ii}^A = I_{ii}^G + m d_i^2 \quad (i = x, y, \text{ or } z)
\]

As noted above, moments of inertia are always positive. It is obvious from the parallel-axes theorem that the minimum moments of inertia of a body occur about axes passing through its mass center. All other inertias must be larger as indicated by the addition of the term “$m d^2$“.
Products of Inertia

The products of inertia of the rigid body are measured relative to a pair of axes and are defined as follows:

\[ I_{xy}^G = I_{yx}^G = \int_B (xy) \, dm \]
\[ I_{xz}^G = I_{zx}^G = \int_B (xz) \, dm \]
\[ I_{yz}^G = I_{zy}^G = \int_B (yz) \, dm \]

Again, the integrals are taken over the entire volume of the mass.

Products of inertia of a body are indicators of symmetry. If a plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero. For example, consider the thin laminate shown. The middle plane of the laminate lies in the XY plane so that half its thickness is above the plane and half is below. Hence, the XY plane is a plane of symmetry and

\[ I_{xz} = I_{yz} = 0 \] (for a thin laminate)

Bodies of revolution have two planes of symmetry. For the configuration shown, the XZ and YZ planes are planes of symmetry. Hence, all products of inertia are zero about the XYZ axis system.

Products of inertia are found by measurement, calculation, or both. Calculations are based on direct integration, the “body build-up” technique, or both. In the body build-up technique, products of inertia of simple shapes are added to estimate the products of inertia of a composite shape. The products of inertia of simple shapes (about their individual mass centers) are found in standard inertia tables. These values are transferred to axes through the composite mass center using the Parallel-Axes Theorem for Products of Inertia.

Parallel-Axes Theorem for Products of Inertia

The product of inertia \( I_{ij}^A \) of a body about a pair of axes \( (i, j) \) passing through any point \( A \) is equal to the sum of the product of inertia \( I_{ij}^G \) of the body about a set of parallel axes through the mass center \( G \) plus the mass \( m \) times the product of the coordinates \( c_i \) and \( c_j \) of \( G \) relative to \( A \) (or \( A \) relative to \( G \)) measured along those axes.

\[ I_{ij}^A = I_{ij}^G + m c_i c_j \quad (i = x, y, \text{ or } z \; \text{and} \; j = x, y, \text{ or } z) \]

Products of inertia can be positive, negative, or zero.
The Inertia Matrix

The moments and products of inertia of a body about a set of axes (passing through some point) can be collected into a single inertia matrix. For example, the inertia matrix of a body about a set of axes passing through its mass center \( G \) is defined as

\[
[I_G] = \begin{bmatrix}
I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\
-I_{xy}^G & I_{yy}^G & -I_{yz}^G \\
-I_{xz}^G & -I_{yz}^G & I_{zz}^G 
\end{bmatrix}
\]

Note that the diagonal entries are the moments of inertia and the off-diagonal entries are the negatives of the products of inertia. Defining the matrix in this way is convenient for calculating the angular momentum of the body as discussed below.

For nonsymmetric bodies, there can be an infinite number of inertia matrices associated with each point of a body because the inertia matrix changes with the orientation of the axes at that point. However, there is generally only one set of axes for each point for which the inertia matrix is diagonal. These axes are called principal axes (or principal directions) and the inertias about those axes are called principal moments of inertia for that point. In general, the principal axes and principal moments of inertia are different for each point of a body.

For symmetric bodies, however, there can be multiple sets of principal axes at a given point and multiple points can have the same principal moments of inertia and principal axes. For example, consider the body of revolution shown in the diagram above. As shown, the \( X \) and \( Y \) axes are principal axes for any point along the \( Z \) axis, and they can be rotated by any angle about the \( Z \) axis to produce another set of principal axes. Of course, any axis passing through the center of a sphere is a principal axis for that point.

All inertia matrices are symmetric. Consequently, they have real eigenvalues and eigenvectors. The eigenvalues of an inertia matrix are the principal moments of inertia and the eigenvectors are the principal directions for that point. If the eigenvectors are normalized, they represent a set of three mutually perpendicular unit vectors in the principal directions.

The principal moments of inertia of a body for some point, say mass-center \( G \), can be calculated by setting

\[
\det \begin{bmatrix}
(I_{xx}^G - \lambda) & -I_{xy}^G & -I_{xz}^G \\
-I_{xy}^G & (I_{yy}^G - \lambda) & -I_{yz}^G \\
-I_{xz}^G & -I_{yz}^G & (I_{zz}^G - \lambda) 
\end{bmatrix} = 0
\]

By expanding the determinant, the resulting characteristic equation can be written as follows.

\[
\lambda^3 + (-I'_{xx} - I'_{yy} - I'_{zz}) \lambda^2 + (I'_{xx}I'_{yy} + I'_{xx}I'_{zz} + I'_{yy}I'_{zz} - I'^2_{xy} - I'^2_{xz} - I'^2_{yz}) \lambda \\
+ (-I'_{xx}I'_{yy}I'_{zz} + I'_{xx}I'^2_{yz} + I'_{xx}I'^2_{xz} + I'_{yy}I'^2_{xz} + I'_{yy}I'^2_{yz} + 2I'_{xy}I'_{xz}I'_{yz}) = 0
\]

The three roots of this equation are the three principal moments of inertia.
If $I_i$ ($i = 1, 2, 3$) represent the three principal moments of inertia, the **principal direction** for each principal moment can be found by writing

$$
\begin{bmatrix}
(I_x' - I) & -I_x' & -I_x' \\
-I_x' & (I_y' - I) & -I_y' \\
-I_y' & -I_y' & (I_z' - I)
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
a_{i2} \\
a_{i3}
\end{bmatrix} = \{0\} \quad (i = 1, 2, 3)
$$

Since the coefficient matrix is *singular*, these equations do **not have a single solution**. The **directions** of the **eigenvectors** are **unique**, but their **magnitudes** are **not**. If the eigenvectors are taken to be **unit vectors**, then $a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1$ and the vectors are unique. To solve for the components of each eigenvector, simply choose a value for one of the components, and then solve for the other two. Finally, **normalize** the resulting vector. The components of these normalized eigenvectors are the direction cosines for the principal directions.

It should be noted here that the process detailed above produces a **unique set of three mutually perpendicular unit eigenvectors** if all the **eigenvalues** are **distinct** (i.e., not equal). As noted above, for **symmetric bodies** that have **principal inertia values** that are **not distinct** (i.e. equal), the **eigenvectors** are **not unique**. However, a set of mutually perpendicular unit eigenvectors can always be found. More details on this topic can be found in the Addendum to this Unit and in reference [3] (R.L. Huston, *Multibody Dynamics*, Butterworth-Heinemann, 1990).

**Inertia matrices** are also **diagonalizable** using their eigenvector (or modal) matrices. The columns of an eigenvector matrix $[M]$ associated with an inertia matrix $[I]$ are formed using the components of the normalized eigenvectors of $[I]$. The diagonal matrix of eigenvalues can then be calculated as follows.

$$
[D] = [M]^T [I] [M]
$$

The eigenvalue associated with an eigenvector appears in the same column of $[D]$ as the eigenvector appears in $[M]$.

**The Inertia Dyadic**

The inertias of a body about a set of axes (passing through some point) can also be collected into a single **inertia dyadic**. For example, the **inertia dyadic** of a body about a set of axes through its mass center $G$ is defined as

$$
I_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \xi_i \xi_j
$$

Here, $\xi_i$ ($i = 1, 2, 3$) are the **unit vectors** directed along the **three axes**, and the **components** of the dyadic are the **nine elements** of the **inertia matrix** $I_{ij}^G$ ($i, j = 1, 2, 3$). The vector products $\xi_i \xi_j$ ($i, j = 1, 2, 3$) are called
This definition makes it clear that each inertia value is associated with a pair of axes. For moments of inertia they are repeated pairs \((x, x), (y, y),\) or \((z, z)\), and for products of inertia they are non-repeated pairs \((x, y), (x, z),\) or \((y, z)\).

### Properties of Dyads

Dyads satisfy many properties. Three very useful properties are

1. \(a \cdot b \neq b \cdot a\)
2. \(c \cdot (a \cdot b) = (c \cdot a)b\) and \((a \cdot b) \cdot c = a(b \cdot c) = (b \cdot c)a\)
3. \((a \cdot b + c \cdot d) \cdot e = (b \cdot e)a + (d \cdot e)c\)

The latter two properties indicate that the “dot” product of a dyad and a vector is a vector. Recall that the “dot” product of two vectors is a scalar. These properties will be used later in the calculation of angular momentum of a body. The dyad-vector dot product is akin to the matrix-vector product of matrix algebra.

### Relationship between Dyadic Components in Different Frames

Like vectors, dyadics can be represented by components in different reference frames. Consider the dyadic \(A\) and its representations in two different reference frames \(B:\(n_1, n_2, n_3\)\) and \(C:\(e_1, e_2, e_3\)\).

\[
A = \sum_{k,\ell=1}^{3} a_{k\ell}^B n_k n_\ell = \sum_{i,j=1}^{3} a_{ij}^C e_i e_j
\]

Here, \(a_{k\ell}^B (k, \ell = 1, 2, 3)\) represent the nine components of \(A\) in \(B:\(n_1, n_2, n_3\)\), and \(a_{ij}^C (i, j = 1, 2, 3)\) represent the nine components of \(A\) in \(C:\(e_1, e_2, e_3\)\). These two sets of components can be related by using the transformation matrix that relates the two reference frames.

If \([R]\) is the matrix that transforms vectors and their components from frame \(C\) into frame \(B\), then

\[
\sum_{i,j} a_{ij}^C e_i e_j = \sum_{i,j} a_{ij}^C \left( \sum_k R_{ik}^C n_k \right) \left( \sum_\ell R_{j\ell}^C n_\ell \right) = \sum_{k,\ell} \left( \sum_{i,j} a_{ij}^C R_{ik}^C R_{j\ell}^T \right) n_k n_\ell = \sum_{k,\ell} a_{k\ell}^B n_k n_\ell
\]

Comparing the last two terms in this equation gives

\[
a_{k\ell}^B = \sum_{i,j} a_{ij}^C R_{ik}^C R_{j\ell}^T = \sum_{i,j} R_{ki} a_{ij}^C R_{j\ell}^T
\]

Note that the sums on indices \(i, j, k,\) and \(\ell\) are all from 1 to 3, and the superscript \(T\) indicates the matrix transpose. The above result can be written in matrix form as

\[
\left[ A^B \right] = [R] \left[ A^C \right] [R]^T
\]
This result can be applied to the inertia matrix of rigid bodies. Given \( I_G^B \) the inertia matrix of a body about a set of axes passing through its mass-center \( G \) and parallel to \( B : (n_1, n_2, n_3) \), \( I_G^C \) the inertia matrix of the body about a second set of axes passing through \( G \) and parallel to \( C : (\ell_1, \ell_2, \ell_3) \) can be calculated as follows

\[
I_G^C = [R] I_G^B [R]^T
\]

As before, \([R]\) transforms vectors and their components from frame \( C \) into \( B \).

**Angular Momentum of a Rigid Body about its Mass Center**

To calculate the angular momentum of a rigid body about its mass center \( G \), consider the rigid body \( B \). The point \( P \) represents an arbitrary point within the body, “\( dm \)” represents the elemental mass of the body associated with \( P \), and \( \mathbf{r}_{P/G} \) represents the position vector of \( P \) with respect to \( G \). The angular momentum of \( B \) about \( G \) is then defined as

\[
H_G = \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_P) \, dm
\]

The integral is taken over the entire volume of the mass.

An alternative form for \( H_G \) can be found by using the kinematic formula for two points fixed on a rigid body and the definition of center of mass (i.e. \( \int_B \mathbf{r}_{P/G} \, dm = 0 \)) as follows

\[
H_G = \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_P) \, dm = \int_B (\mathbf{r}_{P/G} \times (\mathbf{v}_G + \mathbf{v}_{P/G})) \, dm = \left( \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_G) \, dm \right) + \left( \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_{P/G}) \, dm \right)
\]

\[
= \left( \int_B \mathbf{r}_{P/G} \, dm \right) \times \mathbf{v}_G + \left( \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_{P/G}) \, dm \right)
\]

\[
= \int_B (\mathbf{r}_{P/G} \times (\omega_B \times \mathbf{r}_{P/G})) \, dm
\]

\[
\Rightarrow H_G = \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_P) \, dm = \int_B (\mathbf{r}_{P/G} \times \mathbf{v}_{P/G}) \, dm = \int_B (\mathbf{r}_{P/G} \times (\omega_B \times \mathbf{r}_{P/G})) \, dm
\]

This alternative form shows that the angular momentum \( H_G \) incorporates only the angular motion of the body.
A more useful result that specifically relates $H_G$ to the concepts of inertia and angular velocity can be found by letting $\mathbf{r} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3$ and $\mathbf{r}_G = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$, and then using the vector identity 

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$$

to expand the expression for $H_G$. In particular,

$$H_G = \int_B \left( \mathbf{r}_P \times (\mathbf{r}_B \times \mathbf{r}_P) \right) d\mathbf{m} = \int_B \left( \mathbf{r}_P \times \mathbf{r}_P \right) \mathbf{r}_B d\mathbf{m} - \int_B \left( \mathbf{r}_P \times \mathbf{r}_B \right) \mathbf{r}_P d\mathbf{m}$$

$$= \int_B \left( x^2 + y^2 + z^2 \right) \left( \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \right) d\mathbf{m} - \int_B \left( x \omega_1 + y \omega_2 + z \omega_3 \right) \left( x e_1 + y e_2 + z e_3 \right) d\mathbf{m}$$

$$= \int_B r^2 \left( \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \right) d\mathbf{m} - \int_B \left( x \omega_1 + y \omega_2 + z \omega_3 \right) \left( x e_1 + y e_2 + z e_3 \right) d\mathbf{m}$$

Sorting the vector components gives

$$H_G = \int_B \left( r^2 \omega_1 - x \left( x \omega_1 + y \omega_2 + z \omega_3 \right) \right) e_1 d\mathbf{m} + \int_B \left( r^2 \omega_2 - y \left( x \omega_1 + y \omega_2 + z \omega_3 \right) \right) e_2 d\mathbf{m} + \int_B \left( r^2 \omega_3 - z \left( x \omega_1 + y \omega_2 + z \omega_3 \right) \right) e_3 d\mathbf{m}$$

The evaluation of the integrals does not depend on the angular velocity components or the unit vectors, so the above equation can be further simplified as follows.

$$H_G = \left( \omega_1 \int_B \left( y^2 + z^2 \right) d\mathbf{m} + \omega_2 \int_B \left( -x y \right) d\mathbf{m} + \omega_3 \int_B \left( -x z \right) d\mathbf{m} \right) e_1 +$$

$$\left( \omega_1 \int_B \left( -x y \right) d\mathbf{m} + \omega_2 \int_B \left( x^2 + z^2 \right) d\mathbf{m} + \omega_3 \int_B \left( -y z \right) d\mathbf{m} \right) e_2 +$$

$$\left( \omega_1 \int_B \left( -x z \right) d\mathbf{m} + \omega_2 \int_B \left( -y z \right) d\mathbf{m} + \omega_3 \int_B \left( x^2 + y^2 \right) d\mathbf{m} \right) e_3$$

or

$$H_G = \left( I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3 \right) e_1 + \left( -I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3 \right) e_2 + \left( -I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3 \right) e_3$$

Here the integrals are now recognized as the moments and products of inertia of the body about axes parallel to $(e_1, e_2, e_3)$ and passing through the mass center $G$.

Note that for two-dimensional motion the angular momentum of a body is in the same direction as the angular velocity of the body, both being normal to the plane of motion. In three-dimensional motion, however, the angular momentum is generally not in the same direction as the angular velocity. This contrasts with the linear momentum of a body which is in the same direction as the velocity of the mass center of the body for both two and three-dimensional motion.
Representation of Angular Momentum as a Matrix-Vector Product

The above result for the angular momentum vector $\mathbf{H}_G$ is easier to remember when we note the following matrix-vector product can be used to generate the components.

$$
\begin{bmatrix}
\mathbf{H}_G \cdot \mathbf{e}_1 \\
\mathbf{H}_G \cdot \mathbf{e}_2 \\
\mathbf{H}_G \cdot \mathbf{e}_3
\end{bmatrix}
= 
\begin{bmatrix}
I^{G}_{11} & I^{G}_{12} & I^{G}_{13} \\
I^{G}_{21} & I^{G}_{22} & I^{G}_{23} \\
I^{G}_{31} & I^{G}_{32} & I^{G}_{33}
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
= 
\begin{bmatrix}
I^{G}_{xx} & -I^{G}_{xy} & -I^{G}_{xz} \\
-I^{G}_{xy} & I^{G}_{yy} & -I^{G}_{yz} \\
-I^{G}_{xz} & -I^{G}_{yz} & I^{G}_{zz}
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
$$

Here, the inertias and angular velocity components must be resolved (calculated) about the same set of directions in this case indicated by the unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Representation of Angular Momentum as a Dyadic-Vector Product

The angular momentum vector $\mathbf{H}_G$ can also be written as the “dot” product of the inertia dyadic with the angular velocity vector. That is,

$$
\mathbf{H}_G = \mathbf{I}_G \cdot \mathbf{\omega}_B
$$

This is easily verified by substituting for $\mathbf{I}_G$ and $\mathbf{\omega}_B$ in this expression and expanding.

$$
\begin{align*}
\mathbf{H}_G &= \left( \sum_{i=1}^{3} \sum_{j=1}^{3} I^{G}_{ij} \mathbf{e}_i \cdot \mathbf{e}_j \right) \left( \sum_{k=1}^{3} \omega_k \mathbf{e}_k \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \omega_k I^{G}_{ij} \mathbf{e}_i \cdot \mathbf{e}_j \\
&= \left( I^{G}_{11} \omega_1 + I^{G}_{12} \omega_2 + I^{G}_{13} \omega_3 \right) \mathbf{e}_1 + \left( I^{G}_{21} \omega_1 + I^{G}_{22} \omega_2 + I^{G}_{23} \omega_3 \right) \mathbf{e}_2 + \left( I^{G}_{31} \omega_1 + I^{G}_{32} \omega_2 + I^{G}_{33} \omega_3 \right) \mathbf{e}_3 \\
&= \left( I^{G}_{xx} \omega_1 - I^{G}_{xy} \omega_2 - I^{G}_{xz} \omega_3 \right) \mathbf{e}_1 + \left( -I^{G}_{xy} \omega_1 + I^{G}_{yy} \omega_2 - I^{G}_{yz} \omega_3 \right) \mathbf{e}_2 + \left( -I^{G}_{xz} \omega_1 - I^{G}_{yz} \omega_2 + I^{G}_{zz} \omega_3 \right) \mathbf{e}_3
\end{align*}
$$

Here, $\delta_{jk}$ (often called Kronecker’s delta function) is equal to one when $j = k$ and zero when $j \neq k$.

This result is the same as that obtained using the matrix-vector product presented above. However, note that the unit vectors used in the analysis appear explicitly in the dyadic-vector product, whereas they do not in the matrix-vector product. Obviously either approach produces correct results, but care must be taken when using the matrix-vector product to ensure the same directions are used for both the inertia matrix and angular velocity components. The resulting angular momentum components are in the directions of the same set of unit vectors.
Angular Momentum of a Rigid Body about an Arbitrary Point

The angular momentum of a rigid body about an arbitrary point \( A \) is defined as

\[
H_A = \int_B (\mathbf{r}_{P/A} \times \mathbf{v}_P) \, dm
\]

Here, \( \mathbf{r}_{P/A} \) is the position vector of points \( P \) within the body relative to \( A \), and again, the integral is taken over the entire volume of the mass. The angular momentum \( H_A \) can be related to the angular momentum \( H_G \) by recognizing that \( \mathbf{r}_{P/A} = \mathbf{r}_{G/A} + \mathbf{r}_{P/G} \).

Substituting this expression into the integral and expanding gives

\[
H_A = \int_B \left( (\mathbf{r}_{G/A} + \mathbf{r}_{P/G}) \times \mathbf{v}_P \right) \, dm = \int_B \left( \mathbf{r}_{G/A} \times \mathbf{v}_P \right) \, dm + \int_B \left( \mathbf{r}_{P/G} \times \mathbf{v}_P \right) \, dm
\]

\[
= \mathbf{r}_{G/A} \times \left( \int_B \mathbf{v}_P \, dm \right) + \int_B \left( \mathbf{r}_{P/G} \times \mathbf{v}_P \right) \, dm
\]

\[
= \mathbf{r}_{G/A} \times (m \mathbf{v}_G) + H_G
\]

or

\[
H_A = H_G + \mathbf{r}_{G/A} \times m \mathbf{v}_G
\]

The last term in this expression represents the moment of the linear momentum of the body about \( A \) (assuming the line of action of the linear momentum vector passes through \( G \)).

Special Case: Motion about a Fixed Point on the Body

If some point \( O \) of the rigid body is fixed so the body pivots about that point, then the velocity of the mass center can be written as

\[
\mathbf{v}_G = \mathbf{v}_O + \mathbf{\omega}_B \times \mathbf{r}_{G/O} = \mathbf{\omega}_B \times \mathbf{r}_{G/O}.
\]

Substituting this result into the definition for angular momentum gives

\[
H_O = \int_B \left( \mathbf{r}_{P/O} \times \mathbf{v}_P \right) \, dm = \int_B \left( \mathbf{r}_{P/O} \times \mathbf{v}_{P/O} \right) \, dm = \int_B \left( \mathbf{r}_{P/O} \times (\mathbf{\omega}_B \times \mathbf{r}_{P/O}) \right) \, dm
\]

This expression is like that obtained for the mass center except the position vector is referenced to the fixed-point \( O \). Hence, the angular momentum about \( O \) is computed in the same way as for the mass center except the inertia values are measured about \( O \). That is,

\[
H_O = I_{O} \cdot \mathbf{\omega}_B
\]
Here, \( I_O \) is the inertia dyadic (or matrix) about the fixed-point \( O \). If moments and products of inertia are known, then the parallel-axes theorems for moments and products of inertia can be used to compute \( I_O \).

**Kinetic Energy of a Rigid Body**

The figure at the right depicts a rigid body \( B \) moving relative to a fixed frame \( R \). The kinetic energy of \( B \) is defined as

\[
K = \int_B \left( \frac{1}{2} (R_{V_P} \cdot R_{V_P}) \right) dm
\]

Here, \( R_{V_P} \) is the velocity of points \( P \) of the body, and the integral is taken over the entire volume of the mass.

A more useful definition can be derived by relating the velocity of \( P \) to the velocity of the mass center \( G \). Using the relative velocity equation, the integrand can be rewritten as

\[
R_{V_P} \cdot R_{V_P} = (R_{V_P})^2 = (R_{V_G} + (R_{o_B} \times R_{P/G}))^2
\]

Substituting back into the integral gives the following three terms:

1. \[
\int_B \left( \frac{1}{2} (R_{V_G})^2 \right) dm = \frac{1}{2} (R_{V_G})^2 \int_B dm = \frac{1}{2} m (R_{V_G})^2 = \frac{1}{2} m v_G^2
\]

2. \[
\int_B 2 R_{V_G} \cdot (R_{o_B} \times R_{P/G}) dm = 2 R_{V_G} \cdot \left( R_{o_B} \times \left( \int_B R_{P/G} \ dm \right) \right) = 0 \quad \text{... (definition of mass center)}
\]

3. Letting \( R_{P/G} = x e_1 + y e_2 + z e_3 \) and \( R_{o_B} = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \), the integrand of the third integral can be expanded as follows:

\[
\left( R_{o_B} \times R_{P/G} \right)^2 = (\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2
\]

\[
= \omega_1^2 (y^2 + z^2) + \omega_2^2 (x^2 + z^2) + \omega_3^2 (x^2 + y^2) - 2 \omega_1 \omega_2 x y - 2 \omega_1 \omega_3 x z - 2 \omega_2 \omega_3 y z
\]

Substituting into the integral gives

\[
\int_B \left( \frac{1}{2} (R_{o_B} \times R_{P/G})^2 \right) dm = \frac{1}{2} \omega_1^2 \left( \int_B (y^2 + z^2) \right) dm + \frac{1}{2} \omega_2^2 \left( \int_B (x^2 + z^2) \right) dm + \frac{1}{2} \omega_3^2 \left( \int_B (x^2 + y^2) \right) dm
\]

\[
- \omega_1 \omega_2 \left( \int_B x y \right) dm - \omega_1 \omega_3 \left( \int_B x z \right) dm - \omega_2 \omega_3 \left( \int_B y z \right) dm
\]

\[
= \frac{1}{2} \omega_1^2 I_{xx}^G + \frac{1}{2} \omega_2^2 I_{yy}^G + \frac{1}{2} \omega_3^2 I_{zz}^G - \omega_1 \omega_2 I_{xx}^G - \omega_1 \omega_3 I_{xz}^G - \omega_2 \omega_3 I_{yz}^G
\]
It is easy to show that this last result is equal to \( \frac{1}{2} R \mathbf{\omega}_B \cdot \mathbf{I}_G \).

Adding the three terms gives the following result.

\[
K = \frac{1}{2} m \left( \mathbf{v}_G \right)^2 + \frac{1}{2} R \mathbf{\omega}_B \cdot \mathbf{H}_G = \frac{1}{2} m \left( \mathbf{v}_G \right)^2 + \frac{1}{2} R \mathbf{\omega}_B \cdot \mathbf{I}_G \cdot \frac{1}{2} R \mathbf{\omega}_B
\]

**Special Case: Motion about a Fixed-Point \( O \)**

If there is a point \( O \) within the body that is fixed so that the body pivots about \( O \), then

\[
\left( \mathbf{v}_G \right)^2 = \left( \mathbf{v}_G \times \mathbf{L}_{G/O} \right)^2 = \left( y_G^2 + z_G^2 \right) \omega_1^2 + \left( x_G^2 + z_G^2 \right) \omega_2^2 + \left( x_G^2 + y_G^2 \right) \omega_3^2 - 2 \omega_1 \omega_2 x_G y_G - 2 \omega_1 \omega_3 x_G z_G - 2 \omega_2 \omega_3 y_G z_G
\]

Substituting this result into the boxed equation above and combining terms, it can be shown that the kinetic energy can be reduced to purely rotational energy about \( O \).

\[
K = \frac{1}{2} R \mathbf{\omega}_B \cdot \mathbf{H}_O = \frac{1}{2} R \mathbf{\omega}_B \cdot \mathbf{I}_O \cdot \frac{1}{2} R \mathbf{\omega}_B
\]

Here \( \mathbf{I}_O \) is the inertia dyadic (or matrix) for a set of axes passing through the fixed-point \( O \).

**Example 1: Angular Momentum and Kinetic Energy of a Simple Crank Shaft**

The figure shows a simple crank shaft consisting of seven segments, each considered to be a slender bar. Each segment of length \( \ell \) has mass \( m \). There are six segments of length \( \ell \) and one segment of length \( 2\ell \) (segment 4). The mass center of the system \( G \) is located on the axis of rotation.

Reference frames:
- \( R: \mathbf{i}, \mathbf{j}, \mathbf{k} \) (fixed frame)
- \( S: \mathbf{i}', \mathbf{j}', \mathbf{k} \) (rotates with the shaft)

Find:
- a) \( H_G \) the angular momentum of the system about its mass center, \( G \)
- b) \( K \) the kinetic energy of the system

Solution:
a) The elements of the inertia matrix can be found using the parallel-axes theorems for moments and products of inertia and the body build-up technique. However, given that the angular velocity of the system is only about the \( Z \) axis, only the third column of the inertia matrix need be determined. Specifically, the \( S \) frame components of \( H_G \) can be written as follows

\[
\begin{align*}
\{ H_G \cdot \mathbf{i}' \} &= \begin{bmatrix} I_{X'X'}^G & -I_{X'Y'}^G & -I_{X'Z'}^G \end{bmatrix} \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \begin{bmatrix} -I_{X'Z'}^G \omega \end{bmatrix} \\
\{ H_G \cdot \mathbf{j}' \} &= \begin{bmatrix} -I_{Y'X'}^G & I_{Y'Y'}^G & -I_{Y'Z'}^G \end{bmatrix} \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \begin{bmatrix} -I_{Y'Z'}^G \omega \end{bmatrix} \\
\{ H_G \cdot \mathbf{k} \} &= \begin{bmatrix} -I_{Z'X'}^G & -I_{Z'Y'}^G & I_{Z'Z'}^G \end{bmatrix} \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \begin{bmatrix} I_{Z'Z'}^G \omega \end{bmatrix}
\end{align*}
\]
Using the parallel-axes theorem for moments of inertia and the body build-up technique, the moment of inertia of the system about the Z axis can be calculated as follows

\[ I_{zz}^G = \sum_{i=1}^{7} \left( I_{zz}^G \right)_i = 0 + \frac{1}{3}m\ell^2 + m\ell^2 + \frac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \frac{1}{3}m\ell^2 + 0 \Rightarrow I_{zz}^G = \frac{10}{3}m\ell^2 \]

The contributions of each of the seven segments are shown individually in the equation. Segments 1 and 7 lie along the Z-axis and as slender bars have approximately zero inertia about that axis. The Z-axis passes along the ends of segments 2 and 6 so they each contribute \( \frac{1}{3}m\ell^2 \). Segments 3 and 5 are parallel to the Z-axis at a distance of \( \ell \) so they each contribute approximately \( m\ell^2 \), and the Z-axis passes through the mass center of segment 4 so it contributes \( \frac{1}{12}(2m)(2\ell)^2 \).

Since the \( X'Z \) plane is a plane of symmetry, the products of inertia associated with the \( Y' \) direction are zero. Hence, \( I_{xy}^G = 0 \). The product \( I_{xz}^G \), however, is not zero. It can be calculated using the parallel-axes theorem for products of inertia and the body build-up technique as follows

\[ I_{xz}^G = \sum_{i=1}^{7} \left( I_{xz}^G \right)_i = 0 + m\left(\frac{\ell}{2}\right)(-\ell) + m\ell\left(-\frac{\ell}{2}\right) + 0 + m\left(-\ell\right)(\frac{\ell}{2}) + m\left(-\frac{\ell}{2}\right)(\ell) + 0 \Rightarrow I_{xz}^G = -2m\ell^2 \]

Again, the contributions of each of the seven links are shown individually in the equation. To calculate the contribution of each link, imagine a set of local axes passing through the mass centers of each of the segments and parallel to the \( X' \) and \( Z \) axes. (The products of inertia of each of the segments about their local axes are zero due to symmetry.) Then apply the parallel-axes theorem to find the products of inertia about the system’s mass center. For example, the product of inertia of segment 2 about the system’s mass center \( G \) can be calculated as follows

\[
\left( I_{xz}^G \right)_2 = \left( I_{xz}^{G_2} \right)_2 + m\left(\frac{\ell}{2}\right)(-\ell) = -\frac{1}{2}m\ell^2
\]

The first term is the product of inertia of segment 2 about its mass center axes (which, again, is zero due to symmetry) and the second term is the product of “m” times the product of the \( X' \) and \( Z \) coordinates of \( G_2 \) relative to \( G \). Note that the product of the \( X' \) and \( Z \) coordinates of \( G \) relative to \( G_2 \) produces the same result.

A similar approach is taken with each of the segments.

Substituting these results into the expression for \( H_G \) gives

\[ H_G = 2m\ell^2\omega' + \left(\frac{10}{3}m\ell^2\omega_k\right) \]
Note here that even though the **angular motion** is only about the *Z axis*, the **angular momentum** has a component which is **normal to** that direction due to the **mass asymmetry** of the system. Mass asymmetries such as this induce **oscillatory loads** on the support bearings. At significant rotational speeds, these loads cause the supporting structure to **vibrate**. The support loads for this system are calculated in Unit 2 of this volume.

b) The **kinetic energy** of the crank shaft is found from the **velocity** and **angular momentum** vectors to be

\[
K = \frac{1}{2}m(\mathbf{v}_G)^2 + \frac{1}{2}I_G \cdot \mathbf{H}_G = \frac{1}{2}I_G \cdot (\omega \mathbf{k}) \cdot \mathbf{H}_G = \frac{10}{6}m\ell^2\omega^2 \quad \Rightarrow \quad K = \frac{10}{6}m\ell^2\omega^2
\]

From its definition, it is clear the **kinetic energy** of a body incorporates only the **component** of the angular momentum which is in the **direction** of the **angular velocity**.

**Example 2: Angular Momentum and Kinetic Energy of a Misaligned Disk (or Gear)**

The system shown consists of two bodies, shaft *AB* of length *2l* and disk *D* of radius *r*. *D* is **welded** to *AB* so that an axis normal to *D* makes an angle *β* with the shaft axis. A non-zero angle *β* indicates the disk is not aligned properly on the shaft. The shaft and disk rotate together about the *Z* axis at a rate of *ω* (r/s). The mass center of the disk is on the axis of rotation.

Reference frames: 
- *(R)* is the fixed frame
- *(S)*: *i*, *j*, *k* (rotates with the shaft; aligned with the shaft)
- *(D)*: *ε₁*, *ε₂*, *ε₃* (rotates with the shaft; aligned with the disk) \((ε₁ = i')\)

Find:

a) \(H_G\) the **angular momentum** of the disk about its mass center, *G*

b) \(K\) the **kinetic energy** of the disk

Solution:

a) Note the reference frame \((D)\) : *(ε₁*, *ε₂*, *ε₃*) represents a set of **principal axes** for the disk. Assuming the disk is **thin**, its inertia matrix relative to the *D*-frame axes can be written as

\[
[I_G]_D = m r^2 \begin{bmatrix}
\frac{1}{4} & 0 & 0 \\
0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix}
\quad \text{(all products of inertia are zero due to symmetry)}
\]

Here a subscript *D* has been used to indicate the **reference directions** are **fixed** in the disk. To calculate the angular momentum using this result, the components of the angular velocity vector must be resolved in *D* as well. Using \(S_β\) and \(C_β\) to represent the \(\sin(β)\) and \(\cos(β)\), the angular velocity can be written as

\[
\hat{r}_D = \omega \mathbf{k} = \omega (-S_β e_2 + C_β e_3)
\]
Substituting into the definition of angular momentum gives the components of \( H_G \) resolved in \( D \).

\[
\begin{align*}
\{ H_G \cdot \xi_1 \} &= m r^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\omega S_\beta \\ -\omega S_\beta \\ \omega C_\beta \end{bmatrix} = m r^2 \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \omega C_\beta \end{bmatrix} \\
\{ H_G \cdot \xi_2 \} &= m r^2 \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \omega C_\beta \end{bmatrix} \\
\{ H_G \cdot \xi_3 \} &= m r^2 \begin{bmatrix} -\omega S_\beta \\ -\omega S_\beta \\ \omega C_\beta \end{bmatrix} \Rightarrow \{ H_G \} = \frac{1}{4} m r^2 \omega( -S_\beta \xi_2 + 2C_\beta \xi_3 )
\end{align*}
\]

The shaft-based components of \( H_G \) can now be found by recognizing from the diagram that \( \xi_2 = C_\beta \xi'_j - S_\beta \xi'_k \) and \( \xi_3 = S_\beta \xi'_j + C_\beta \xi'_k \). Substituting gives

\[
\begin{align*}
H_G &= \frac{1}{4} m r^2 \omega \left( -S_\beta \xi_2 + 2C_\beta \xi_3 \right) = \frac{1}{4} m r^2 \omega \left[ -S_\beta \left( C_\beta \xi'_j - S_\beta \xi'_k \right) + 2C_\beta \left( S_\beta \xi'_j + C_\beta \xi'_k \right) \right] \\
\text{or} \quad H_G &= \frac{1}{4} m r^2 \omega \left[ (S_\beta C_\beta) \xi'_j + (2C_\beta^2 + S_\beta^2) \xi'_k \right] = \frac{1}{4} m r^2 \omega \left[ (S_\beta C_\beta) \xi'_j + (C_\beta^2 + 1) \xi'_k \right]
\end{align*}
\]

As with the system of Example 1, the angular momentum has a component which is normal to the angular velocity due to the mass asymmetry of the system. If the misalignment angle is set to zero, \( H_G \) reverts to a simpler form which is in the direction of the angular velocity.

\[
H_G = \frac{1}{4} m r^2 \omega \left[ (S_\beta C_\beta) \xi'_j + (2C_\beta^2 + S_\beta^2) \xi'_k \right] = \frac{1}{4} m r^2 \omega \left[ (S_\beta C_\beta) \xi'_j + (C_\beta^2 + 1) \xi'_k \right] = \frac{1}{4} m r^2 \omega \left( 2 \xi'_k \right) \Rightarrow H_G = \frac{1}{2} m r^2 \omega \xi'_k
\]

Note here that the angular momentum was calculated about the shaft-based system without first finding the inertias about those axes. However, the above result can be used to determine these inertias by noting that

\[
H_G = \left( -I_{XZ} \omega \right) \xi'_j + \left( -I_{YZ} \omega \right) \xi'_k + \left( I_{ZZ} \omega \right) k = \frac{1}{4} m r^2 \omega \left[ (S_\beta C_\beta) \xi'_j + (C_\beta^2 + 1) \xi'_k \right]
\]

Equating each of the vector components leads to the following inertias about the shaft axes.

\[
I_{XZ} = 0, \quad I_{YZ} = -\frac{1}{4} m r^2 S_\beta C_\beta, \quad I_{ZZ} = \frac{1}{4} m r^2 \left( C_\beta^2 + 1 \right)
\]

b) The kinetic energy of the disk is found from the velocity and angular momentum vectors to be

\[
K = \frac{1}{2} m \left( \frac{\omega \xi_2}{v_G} \right)^2 + \frac{1}{2} m \left( \frac{\omega \xi_2}{v_D} \right)^2 + \frac{1}{2} m \left( \frac{\omega \xi_2}{H} \right)^2 = \frac{1}{2} \omega^2 k \cdot H_G = \frac{1}{2} \omega^2 \left( 2 \xi'_k \right) \Rightarrow K = \frac{1}{8} m r^2 \left( C_\beta^2 + 1 \right) \omega^2
\]

Again, note that the kinetic energy involves only the component of the angular momentum in the direction of the angular velocity.
Example 3: Angular Momentum and Kinetic Energy of a Rotating Bar

The system shown consists of two bodies, the frame \( F \) and the bar \( B \). Frame \( F \) rotates about the fixed vertical direction annotated by the unit vector \( \vec{k} \). Bar \( B \) is pinned to and rotates about the horizontal arm of \( F \). \( F \) rotates relative to the ground at a rate \( \Omega \) \((r/s)\) and \( B \) rotates relative to \( F \) at a rate of \( \omega \) \((r/s)\).

Reference frames: (\( R \) is the fixed frame)

- \( F : n_1, n_2, k \) (rotates with frame \( F \))
- \( B : e_1, n_2, e_3 \) (rotates with the bar \( B \))

Find:

a) \( H_G \) the angular momentum of \( B \) about its mass center, \( G \)

b) \( K \) the kinetic energy of \( B \)

Solution:

Assuming the bar is slender, the inertia matrix of the bar about its mass center associated with frame \( B \) can be written as follows. Note again that the subscript \( B \) indicates the reference directions are fixed in \( B \).

\[
[I_G]_B = \frac{1}{12} m \ell^2 \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

To use this result to find \( H_G \), the angular velocity must also be resolved into body-fixed components. Using the summation rule for angular velocities and \( S_\theta \) and \( C_\theta \) to represent the \( \sin(\theta) \) and \( \cos(\theta) \), the angular velocity can be written as

\[
\omega_B = \omega_F + \Omega k = \omega n_2 + \Omega (-S_\theta e_1 + C_\theta e_3)
\]

The body-fixed components of the angular momentum vector can then be calculated as follows.

\[
\begin{bmatrix}
H_G \cdot e_1 \\
H_G \cdot n_2 \\
H_G \cdot e_3
\end{bmatrix} = \frac{m \ell^2}{12} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
-S_\theta \\
\omega \\
C_\theta
\end{bmatrix} = \frac{m \ell^2}{12} \begin{bmatrix}
0 \\
\omega \\
\Omega C_\theta
\end{bmatrix}
\]

\[
H_G = \frac{m \ell^2}{12} (\omega n_2 + \Omega C_\theta e_3)
\]

b) The kinetic energy of the bar includes both translational and rotational energy. Using the angular velocity and angular momentum vectors and noting that \( r_{\nu_G} = -d \Omega n_1 \), the kinetic energy can now be written as

\[
K = \frac{1}{2} m(r_{\nu_G})^2 + \frac{1}{2} \omega_B \cdot H_G = \frac{1}{2} m \ell^2 \Omega^2 + \frac{1}{2} (-S_\theta e_1 + \omega n_2 + \Omega C_\theta e_3) \cdot \frac{m \ell^2}{12} (\omega n_2 + \Omega C_\theta e_3)
\]

\[
\Rightarrow K = \frac{1}{2} m \ell^2 \Omega^2 + \frac{m \ell^2}{24} (\omega^2 + \Omega^2 C_\theta^2)
\]


**Example 4: Aircraft with Two Engines**

The aircraft shown has two engines, one on each wing. The orientation of the aircraft relative to a fixed reference frame \( R \) is defined by a 3-2-1 body-fixed rotation sequence \((\psi, \theta, \phi)\). For the purposes of this example, the aircraft is made up of three main components, the **airframe** \( A \) and the two engines \( E_1 \) and \( E_2 \). The term **airframe** is used to refer to all the stationary components of the aircraft, and the term **engine** is used to refer to the rotating components of the engines. The points \( G_i \) (\( i = 1, 2 \)) are the mass centers of the two engines, \( G_A \) is the mass center of the airframe, and \( G \) is the mass center of the aircraft.

The aircraft is symmetrical with respect to the \( b_b z_b \) plane. The two engines are assumed to be identical and placed symmetrically on the airframe so the position vectors of \( G_i \) (\( i = 1, 2 \)) relative to \( G \) the mass center of the aircraft can be written as \( L_{G_i/G} = x_E b_1 + y_E b_2 + z_E b_3 \) and \( L_{G_2/G} = x_E b_1 - y_E b_2 + z_E b_3 \). The engines (rotating components) are assumed to be solids of revolution aligned with the \( b_b \) axis (meaning they are rotationally symmetrical about that axis). Finally, the velocity of the mass center of the aircraft is given in body-frame as \( ^R v_{G} = u b_1 + v b_2 + w b_3 \) and in the ground-frame as \( ^R v_{G} = \dot{X} N_1 + \dot{Y} N_2 + \dot{Z} N_3 \).

**Reference frames:**
- \( R : N_1, N_2, N_3 \) (inertial or ground frame)
- \( A : b_1, b_2, b_3 \) (frame fixed in the aircraft)

**Find:** (express vector components in frame \( A \))

a) \( H_{G_A} \) the angular momentum of the airframe about its mass center \( G_A \)

b) \( H_{G_i} \) the angular momenta of the engines about their mass centers \( G_i \) (\( i = 1, 2 \))

c) \( H_G \) the angular momentum of the aircraft (airframe and engines) about its mass center \( G \)

d) \( K \) the kinetic energy of the aircraft

**Solution:**

a) Assuming the \( x_b z_b \) plane is a plane of symmetry of the airframe, its inertia matrix about its mass center \( G_A \) can be written as

\[
\begin{align*}
I_{G_A} &= \begin{bmatrix}
I_{x_b x_b}^{G_A} & -I_{y_b x_b}^{G_A} & -I_{z_b x_b}^{G_A} \\
-I_{y_b x_b}^{G_A} & I_{y_b y_b}^{G_A} & -I_{z_b y_b}^{G_A} \\
-I_{z_b x_b}^{G_A} & -I_{z_b y_b}^{G_A} & I_{z_b z_b}^{G_A}
\end{bmatrix} = \begin{bmatrix}
I_{x_b x_b}^{G_A} & 0 & -I_{y_b x_b}^{G_A} \\
0 & I_{y_b y_b}^{G_A} & 0 \\
-I_{y_b x_b}^{G_A} & 0 & I_{z_b z_b}^{G_A}
\end{bmatrix}
\end{align*}
\]
Recall the subscript A on the inertia matrix indicates its elements are measured about \textit{airframe-fixed axes}, and note that because the aircraft is symmetrical about the \(x_bz_b\) plane, \(I_{x_bz_b}^G = I_{y_bz_b}^G = 0\). In Unit 5 of Volume I the \textit{angular velocity} of a body whose orientation is described using a 3-2-1, body-fixed orientation angle sequence was found to be

\[
\vec{\omega}_A = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 = \left( \dot{\phi} - \dot{\psi} S_\phi \right) \mathbf{b}_1 + \left( \dot{\theta} C_\phi + \dot{\psi} C_\phi S_\phi \right) \mathbf{b}_2 + \left( -\dot{\theta} S_\phi + \dot{\psi} C_\phi C_\phi \right) \mathbf{b}_3
\]

The \textit{body-fixed components} of the \textit{angular momentum} of the airframe can then be calculated as follows

\[
\begin{align*}
\begin{pmatrix}
H_{G_i} \cdot \mathbf{b}_1 \\
H_{G_i} \cdot \mathbf{b}_2 \\
H_{G_i} \cdot \mathbf{b}_3
\end{pmatrix} &= \begin{pmatrix}
-I_{x_b}^G & 0 & -I_{x_b}^G \\
0 & I_{y_b}^G & 0 \\
-I_{y_b}^G & 0 & I_{z_b}^G
\end{pmatrix} \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} = \begin{pmatrix}
I_{x_b}^G \omega_1 - I_{x_b}^G \omega_3 \\
I_{y_b}^G \omega_2 \\
-I_{y_b}^G \omega_1 + I_{z_b}^G \omega_3
\end{pmatrix}
\end{align*}
\]

\[
\Rightarrow H_{G_i} = \left( I_{x_b}^G \omega_1 - I_{x_b}^G \omega_3 \right) \mathbf{b}_1 + \left( I_{y_b}^G \omega_2 \right) \mathbf{b}_2 + \left( -I_{y_b}^G \omega_1 + I_{z_b}^G \omega_3 \right) \mathbf{b}_3
\]

b) Given the \textit{rotating components} of the engines are \textit{solids of revolution} whose axes are parallel to \(\mathbf{b}_1\), the \textit{inertia matrices} of the engines about a set of axes parallel to reference frame \(A\) and passing through the mass centers of the engines can be written as

\[
I_{G_i}^E = \begin{pmatrix}
-I_{x_b}^E & 0 & -I_{x_b}^E \\
0 & I_{y_b}^E & 0 \\
-I_{y_b}^E & 0 & I_{z_b}^E
\end{pmatrix}
\]

\[
(i = 1, 2)
\]

Note here that the inertias \(I_{y_b}^E\) and \(I_{z_b}^E\) are \textit{equal} and they are \textit{constant} relative to directions fixed in the airframe \(A\) because of the assumed \textit{rotational symmetry} about the \(x_b\) axis.

The \textit{angular velocities} of the engines can be calculated using the \textit{summation rule} for angular velocities.

\[
\vec{\omega}_{E_i} = \vec{\omega}_A + \vec{\omega}_{E_i} = \left( \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 \right) + \left( \omega_{E_i} \mathbf{b}_1 \right)
\]

\[
\Rightarrow \vec{\omega}_{E_i} = \left( \omega_1 + \omega_{E_i} \right) \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 \quad (i = 1, 2)
\]

Using the above results, the \textit{aircraft-fixed components} of the \textit{angular momenta} of the engines can then be calculated as follows

\[
\begin{align*}
\begin{pmatrix}
H_{G_i} \cdot \mathbf{b}_1 \\
H_{G_i} \cdot \mathbf{b}_2 \\
H_{G_i} \cdot \mathbf{b}_3
\end{pmatrix} &= \begin{pmatrix}
I_{x_b}^E & 0 & 0 \\
0 & I_{y_b}^E & 0 \\
0 & 0 & I_{z_b}^E
\end{pmatrix} \begin{pmatrix}
\omega_1 + \omega_{E_i} \\
\omega_2 \\
\omega_3
\end{pmatrix} = \begin{pmatrix}
I_{x_b}^E \left( \omega_1 + \omega_{E_i} \right) \\
I_{y_b}^E \omega_2 \\
I_{z_b}^E \omega_3
\end{pmatrix}
\end{align*}
\]

\[
\Rightarrow H_{G_i} = \left( I_{x_b}^E \left( \omega_1 + \omega_{E_i} \right) \right) \mathbf{b}_1 + \left( I_{y_b}^E \omega_2 \right) \mathbf{b}_2 + \left( I_{z_b}^E \omega_3 \right) \mathbf{b}_3 \quad (i = 1, 2)
\]
c) The angular momentum of the aircraft about its mass center $G$ is the sum of the angular momenta of the airframe and the two engines about $G$.

$$H_G = (H_G)_A + \sum_{i=1}^{2} (H_G)_{E_i}$$

The angular momentum of the airframe about the mass center of the aircraft can be calculated as follows.

$$(H_G)_A = H_{G_A} + (\mathbf{r}_{G_A/G} \times m_A \mathbf{Y}_{G_A}) = H_{G_A} + (\mathbf{r}_{G_A/G} \times m_A (R_{Y_{G_A}} + R_{Y_{G_A/G}}))$$

$$= H_{G_A} + (\mathbf{r}_{G_A/G} \times m_A R_{Y_{G_A}}) + (m_A R_{Y_{G_A}})$$

The second term on the right side can be expanded using the vector identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ and letting $\mathbf{r}_{G_A/G} = x_A \mathbf{b}_1 + z_A \mathbf{b}_3$ gives

$$m_A (\mathbf{r}_{G_A/G} \times (R_{\mathbf{w}_A} \times R_{\mathbf{r}_{G_A/G}})) = m_A \left( (\mathbf{r}_{G_A/G} \cdot R_{\mathbf{w}_A}) R_{\mathbf{w}_A} - (\mathbf{r}_{G_A/G} \cdot R_{\mathbf{w}_A}) R_{\mathbf{r}_{G_A/G}} \right)$$

$$= m_A \left[ \left( x_A^2 + z_A^2 \right) \mathbf{w}_1 - \left( x_A^2 \right) \mathbf{w}_1 - \left( z_A^2 \right) \mathbf{w}_2 \right] \mathbf{b}_2 + m_A \left[ \left( x_A^2 + z_A^2 \right) \mathbf{w}_2 + \left( x_A^2 \right) \mathbf{w}_2 - \left( z_A^2 \right) \mathbf{w}_3 \right] \mathbf{b}_3$$

$$\Rightarrow m_A (\mathbf{r}_{G_A/G} \times (R_{\mathbf{w}_A} \times R_{\mathbf{r}_{G_A/G}})) = m_A \left[ \left( x_A^2 \right) \mathbf{w}_1 - \left( x_A z_A \right) \mathbf{w}_2 \right] \mathbf{b}_2 + m_A \left[ \left( x_A^2 + z_A^2 \right) \mathbf{w}_2 + \left( x_A^2 \right) \mathbf{w}_2 - \left( z_A^2 \right) \mathbf{w}_3 \right] \mathbf{b}_3$$

Note that the $b_2$ component of the position vector $\mathbf{r}_{G_A/G}$ is zero because the engines are symmetrically placed with respect to the $x_b z_b$ plane.

Combining this term with $H_{G_A}$ and using the parallel-axes theorems for moments and products of inertia gives

$$H_{G_A} + m_A (\mathbf{r}_{G_A/G} \times (R_{\mathbf{w}_A} \times R_{\mathbf{r}_{G_A/G}}))$$

$$= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} I_{x_b y_b}^{G_A} + m_A z_A^2 & 0 & -I_{y_b z_b}^{G_A} + m_A x_A z_A \\ 0 & I_{y_b y_b}^{G_A} + m_A (x_A^2 + z_A^2) & 0 \\ -I_{x_b z_b}^{G_A} + m_A x_A z_A & 0 & I_{z_b z_b}^{G_A} + m_A x_A^2 \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} (I_{x_b y_b}^{G_A})_A & 0 & -(I_{y_b z_b}^{G_A})_A \\ 0 & (I_{y_b y_b}^{G_A})_A & 0 \\ -(I_{x_b z_b}^{G_A})_A & 0 & (I_{z_b z_b}^{G_A})_A \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix}$$
Here, \( (I_G)_A \) represents the \textit{inertia tensor} of the \textit{airframe} as measured about \( G \) the mass center of the aircraft.

Substituting this result into the original expression for \( (H_G)_A \) gives

\[
(H_G)_A = (I_G)_A \cdot \vec{r}_A + (m_A \vec{r}_{G_A/G}) \times \vec{y}_G
\]

Using a \textit{similar procedure} for each of the engines along with the \textit{summation rule} for angular velocities, the expressions for the angular momenta of the engines can be simplified as follows.

\[
(H_G)_{E_i} = H_{G_{E_i}} + \left( \vec{r}_{G_{E_i}/G} \times m_E \vec{y}_{G_{E_i}/G} \right) + \left( \vec{r}_{G_{E_i}/G} \times m_E \vec{y}_{G_{E_i}/G} \right) + \left( \vec{r}_{G_{E_i}/G} \times m_E \vec{y}_{G_{E_i}/G} \right)
\]

\[
= \left( I_{G_{E_i}} \cdot \vec{r}_A + (m_E \vec{r}_{G_{E_i}/G}) \times \vec{y}_G \right) + \left( I_{G_{E_i}} \cdot \vec{A}_E \right) + \left( m_E \vec{r}_{G_{E_i}/G} \right) \times \vec{y}_G
\]

\[
\Rightarrow (H_G)_{E_i} = \left( I_{G_{E_i}} \cdot \vec{r}_A \right) + \left( I_{G_{E_i}} \cdot \vec{A}_E \right) + \left( m_E \vec{r}_{G_{E_i}/G} \right) \times \vec{y}_G
\]

Substituting all terms into the equation for \( H_G \) gives

\[
H_G = (H_G)_A + \sum_{i=1}^{2} (H_G)_{E_i}
\]

\[
= \left( I_{G_{aircraft}} \cdot \vec{r}_A \right) + \left( (m_A \vec{r}_{G_{aircraft}/G}) \times \vec{y}_G \right) + \left( I_{G_{E_1}} \cdot \vec{r}_A \right) + \left( I_{G_{E_1}} \cdot \vec{A}_E \right) + \left( m_E \vec{r}_{G_{E_1}/G} \right) \times \vec{y}_G
\]

\[
\Rightarrow H_G = \left( I_{G_{aircraft}} \cdot \vec{r}_A \right) + \left( I_{G_{E_1}} \cdot \vec{A}_E \right) + \left( m_E \vec{r}_{G_{E_1}/G} \right) \times \vec{y}_G
\]

Here, \( (I_G)_{aircraft} \) represents the inertia tensor of the entire aircraft about its mass center \( G \), and using the definition of \textit{center of mass}, the sum \( m_A \vec{r}_{G_{aircraft}/G} + m_E \vec{r}_{G_{E_1}/G} \) is recognized to be \textit{zero}. A more specific \textit{form} for the \textit{airframe-fixed components} of \( H_G \) can be written as follows.
\[
\begin{align*}
\begin{bmatrix}
H_G \cdot b_1 \\
H_G \cdot b_2 \\
H_G \cdot b_3
\end{bmatrix} &= \begin{bmatrix}
I^G_{\nu_{x_0}} & 0 & -I^G_{\nu_{y_0}} \\
0 & I^G_{\nu_{y_3}} & 0 \\
-I^G_{\nu_{x_0}} & 0 & I^G_{\nu_{y_3}}
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
+ \begin{bmatrix}
I^E_{\nu_{x_0}} & 0 & 0 \\
0 & I^E_{\nu_{y_3}} & 0 \\
0 & 0 & I^E_{\nu_{y_3}}
\end{bmatrix}
\begin{bmatrix}
\omega_{E_1} \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
I^E_{\nu_{x_0}} & 0 & 0 \\
0 & I^E_{\nu_{y_3}} & 0 \\
0 & 0 & I^E_{\nu_{y_3}}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

\[
\Rightarrow \begin{bmatrix}
H_G \cdot b_1 \\
H_G \cdot b_2 \\
H_G \cdot b_3
\end{bmatrix} = \begin{bmatrix}
I^G_{\nu_{x_0}} \omega_1 - I^G_{\nu_{y_0}} \omega_3 + I^E_{\nu_{x_3}} \omega_{E_1} + I^E_{\nu_{y_3}} \omega_{E_2} \\
I^G_{\nu_{y_3}} \omega_2 \\
-I^G_{\nu_{x_0}} \omega_1 + I^G_{\nu_{y_0}} \omega_3
\end{bmatrix}
\]

Note here that \( I^G_{ij} \) (or \( I^E_{ij} \)) represent moments and products of inertia of the entire aircraft about its mass center \( G \) while \( I^E_{x_3,y_3} \) represents the moments of inertia of just the rotating components of the engines about their axes of rotation.

d) The kinetic energy of the aircraft is the sum of the kinetic energies of the airframe and its two engines.

\[
K = K_A + \sum_{i=1}^{2} K_{E_i}
= \left( \frac{1}{2} m_A \left( R_{\nu_{G,A}} \right)^2 + \frac{1}{2} R_{\nu_{A}} \cdot H_G \right) + \sum_{i=1}^{2} \left( \frac{1}{2} m_E \left( R_{\nu_{G_i}} \right)^2 + \frac{1}{2} R_{\nu_{E_i}} \cdot H_{G_i} \right)
\]

The above expression has three translational kinetic energy terms and three rotational kinetic energy terms. It can be transformed into an expression with a single translational energy term associated with \( G \) the mass center of the aircraft as follows. First, rewrite the translational energy of the airframe and the engines in terms of \( R_{\nu_{G}} \).

\[
\frac{1}{2} m_A \left( R_{\nu_{G_A}} \right)^2 = \frac{1}{2} m_A \left( R_{\nu_{G}} + R_{\nu_{G_A/G}} \right)^2 = \frac{1}{2} m_A \left( R_{\nu_{G}} + \left( R_{\nu_{A}} \times R_{G_A/G} \right) \right)^2
= \frac{1}{2} m_A R_{\nu_{G}}^2 + \frac{1}{2} m_A \left( R_{\nu_{A}} \times R_{G_A/G} \right)^2
+ R_{\nu_{G}} \cdot \left( R_{\nu_{A}} \times m_A R_{G_A/G} \right)
\]

Similarly, for the translational kinetic energies of the engines

\[
\frac{1}{2} m_E \left( R_{\nu_{G_i}} \right)^2 = \frac{1}{2} m_E \left( R_{\nu_{G}} + R_{\nu_{G_i/G}} \right)^2 = \frac{1}{2} m_E \left( R_{\nu_{G}} + \left( R_{\nu_{A}} \times R_{G_i/G} \right) \right)^2
= \frac{1}{2} m_E R_{\nu_{G}}^2 + \frac{1}{2} m_E \left( R_{\nu_{A}} \times R_{G_i/G} \right)^2
+ R_{\nu_{G}} \cdot \left( R_{\nu_{A}} \times m_E R_{G_i/G} \right)
\]

(i = 1, 2)

Summing these three terms gives
\[
\frac{1}{2} m_A \left( R_{V_G A} \right)^2 + \sum_{i=1}^{2} \frac{1}{2} m_E \left( R_{V_{iG}} \right)^2 \\
= \frac{1}{2} m_A \left( R_{V_G} \right)^2 + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 + R_{\omega_G} \cdot \left( R_{\omega_A \times m_A L_{G,A/G}} \right) \\
+ \sum_{i=1}^{2} \left[ \frac{1}{2} m_E \left( R_{V_i G} \right)^2 + \frac{1}{2} m_E \left( R_{\omega_i A \times L_{G,i/G}} \right)^2 + R_{\omega_i G} \cdot \left( R_{\omega_i A \times m_i E L_{G,i/G}} \right) \right] \\
= \frac{1}{2} \left( m_A + 2 m_E \right) \left( R_{V_G} \right)^2 + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 + \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,1/G}} \right)^2 + \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,2/G}} \right)^2 \\
+ R_{\omega_G} \cdot \left[ R_{\omega_A \times \left( m_A L_{G,A/G} + m_E L_{G,1/G} + m_E L_{G,2/G} \right)} \right] \\
\quad \text{zero}
\]

\[
\frac{1}{2} m_A \left( R_{V_G A} \right)^2 + \sum_{i=1}^{2} \frac{1}{2} m_E \left( R_{V_{iG}} \right)^2 = \frac{1}{2} m_T \left( R_{V_G} \right)^2 + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 \\
+ \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,1/G}} \right)^2 + \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,2/G}} \right)^2
\]

Here, \( m_T = m_A + 2 m_E \) is the total mass of the aircraft and, using the definition of center of mass, the sum in square brackets is recognized to be zero. Finally, substituting this result into the original expression for \( K \) gives

\[
K = \frac{1}{2} m_T \left( R_{V_G} \right)^2 + \left[ \frac{1}{2} R_{\omega_A \cdot H_{G_A}} + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 \right] \\
+ \left[ \frac{1}{2} R_{\omega_{E1} \cdot H_{G1}} + \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,1/G}} \right)^2 \right] + \left[ \frac{1}{2} R_{\omega_{E2} \cdot H_{G2}} + \frac{1}{2} m_E \left( R_{\omega_A \times L_{G,2/G}} \right)^2 \right]
\]

The three terms in square brackets on the right side of this result can be further simplified as follows. Consider the first bracketed term associated with the airframe and recall the vector identity \((a \times b) \cdot c = a \cdot (b \times c)\).

\[
\frac{1}{2} R_{\omega_A \cdot H_{G_A}} + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 = \frac{1}{2} R_{\omega_A \cdot H_{G_A}} + \frac{1}{2} m_A \left( R_{\omega_A \cdot L_{G,A/G}} \right) \cdot \left( R_{\omega_A \times L_{G,A/G}} \right)
\]

\[
= \frac{1}{2} R_{\omega_A \cdot H_{G_A}} + \frac{1}{2} m_A R_{\omega_A} \cdot \left( L_{G,A/G} \times \left( R_{\omega_A \times L_{G,A/G}} \right) \right)
\]

\[
= \frac{1}{2} R_{\omega_A} \left[ H_{G_A} + m_A \left( R_{\omega_A \times L_{G,A/G}} \right) \right]
\]

Using the results found above for the term in square brackets gives

\[
\frac{1}{2} R_{\omega_A \cdot H_{G_A}} + \frac{1}{2} m_A \left( R_{\omega_A \times L_{G,A/G}} \right)^2 = \frac{1}{2} R_{\omega_A} \cdot \left( L_{G,A/G} \right) \cdot R_{\omega_A}
\]

Following a similar process for the engines along with the summation rule for angular velocities gives
\[ \frac{1}{2} R \omega_{E_i} \cdot H_{G_i} + \frac{1}{2} m_E \left( R \omega_B \times \mathbf{I}_{G,G_i} \right)^2 = \frac{1}{2} \left( R \omega_A + A \omega_{E_i} \right) \cdot H_{G_i} + \frac{1}{2} m_E \left( R \omega_A \times \mathbf{I}_{G,G_i} \right) \left( R \omega_A \times \mathbf{I}_{G,G_i} \right) \]

\[ = \frac{1}{2} R \omega_A \cdot \left( I_{G_i} \cdot \left( R \omega_A + A \omega_{E_i} \right) \right) + \frac{1}{2} A \omega_{E_i} \cdot \left( I_{G_i} \cdot \left( R \omega_A + A \omega_{E_i} \right) \right) + \frac{1}{2} m_E R \omega_A \cdot \left( \mathbf{I}_{G,G_i} \times \left( R \omega_A \times \mathbf{I}_{G,G_i} \right) \right) \]

\[ = \frac{1}{2} \left( R \omega_A \cdot \left( I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} m_E R \omega_A \cdot \left( \mathbf{I}_{G,G_i} \times \left( R \omega_A \times \mathbf{I}_{G,G_i} \right) \right) \right) + \frac{1}{2} \left( R \omega_A \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]

\[ + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]

Again, using results found above, write

\[
\Rightarrow \quad \frac{1}{2} R \omega_{E_i} \cdot H_{G_i} + \frac{1}{2} m_E \left( R \omega_A \times \mathbf{I}_{G,G_i} \right)^2
\]

\[ = \frac{1}{2} \left( R \omega_A \cdot \left( I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} \left( R \omega_A \cdot I_{G_i} \cdot A \omega_{E_i} \right) \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]

Substituting into the kinetic energy function and simplifying gives

\[ K = \frac{1}{2} m_T \left( R \mathbf{v}_G \right)^2 + \frac{1}{2} \left( R \omega_A \cdot H_{G_A} + \frac{1}{2} m_A \left( R \omega_A \times \mathbf{I}_{G_A,G} \right)^2 \right) \]

\[ + \left( \frac{1}{2} R \omega_{E_1} \cdot H_{G_1} + \frac{1}{2} m_E \left( R \omega_A \times \mathbf{I}_{G,G_i} \right)^2 \right) + \left( \frac{1}{2} R \omega_{E_2} \cdot H_{G_2} + \frac{1}{2} m_E \left( R \omega_A \times \mathbf{I}_{G,G_i} \right)^2 \right) \]

\[ = \frac{1}{2} m_T \left( R \mathbf{v}_G \right)^2 + \frac{1}{2} \left[ R \omega_A \cdot \left( I_{G_i} \cdot R \omega_A \right) \right] \]

\[ + \frac{1}{2} \left( R \omega_{E_1} \cdot I_{E_i} \cdot R \omega_A \right) + \frac{1}{2} \left( R \omega_{E_2} \cdot I_{E_2} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]

\[ + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]

\[ K = \frac{1}{2} m_T \left( R \mathbf{v}_G \right)^2 + \frac{1}{2} \left[ R \omega_A \cdot \left( I_{G_i} \cdot R \omega_A \right) \right] \]

\[ + \frac{1}{2} \left( R \omega_{E_1} \cdot I_{E_i} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot R \omega_A \right) + \frac{1}{2} \left( A \omega_{E_i} \cdot I_{G_i} \cdot A \omega_{E_i} \right) \]
This general expression above can be reduced to a more specific result as follows. Considering each term individually:

\[
\frac{1}{2} m_T \left( R^2 \right) = \frac{1}{2} m_T \left( u^2 + v^2 + w^2 \right) = \frac{1}{2} m_T \left( \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right)
\]

\[
\left( \frac{I}{\omega} \right)_{\text{aircraft}} = \begin{bmatrix}
I^G_{x,y} & 0 & -I^G_{x,z} \\
0 & I^G_{y,z} & 0 \\
-I^G_{x,z} & 0 & I^G_{z,y}
\end{bmatrix}
\]

\[
\frac{1}{2} R^A \cdot \left( I^G \right)_{\text{aircraft}} \cdot \frac{R}{\omega} = \frac{1}{2} R^A \cdot \left[ \left( I^G_{x,y} \omega_1 - I^G_{x,z} \omega_3 \right) b_1 + \left( I^G_{y,z} \omega_2 \right) b_2 + \left( -I^G_{z,y} \omega_1 + I^G_{z,x} \omega_3 \right) b_3 \right]
\]

\[
\Rightarrow \frac{1}{2} R^A \cdot \left( I^G \right)_{\text{aircraft}} \cdot \frac{R}{\omega} = \frac{1}{2} \left[ \left( I^G_{x,y} \omega_1 + I^G_{y,z} \omega_2 + I^G_{z,x} \omega_3 - 2I^G_{x,z} \omega_1 \omega_3 \right) \right]
\]

Substituting into the general expression for \( K \) gives the final detailed result.

\[
K = \frac{1}{2} m_T \left( u^2 + v^2 + w^2 \right) + \frac{1}{2} \left[ I^G_{x,y} \omega_1^2 + I^G_{y,z} \omega_2^2 + I^G_{z,x} \omega_3^2 - 2I^G_{x,z} \omega_1 \omega_3 \right]
\]

Or,

\[
K = \frac{1}{2} m_T \left( \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right) + \frac{1}{2} \left[ I^G_{x,y} \omega_1^2 + I^G_{y,z} \omega_2^2 + I^G_{z,x} \omega_3^2 - 2I^G_{x,z} \omega_1 \omega_3 \right]
\]

Note (as before) that \( I^G_{ij} \) (i.e., \( i, j = x, y, \) or \( z \)) represent moments and products of inertia of the entire aircraft about its mass center \( G \) while \( I^E_{x,y} \) represents the moments of inertia of just the rotating components of the engines about their axes of rotation. Also, recall that although both engines are identical, they may be rotating at different speeds.
Example 5: Double Pendulum or Arm

The system shown is a three-dimensional double pendulum or arm. The first link is connected to ground and the second link is connected to the first with ball and socket joints at $O$ and $A$. The orientation of each link is defined relative to the ground using a 3-1-3 body-fixed rotation sequence. The lengths of the links are $\ell_1$ and $\ell_2$. The links are assumed to be slender bars with mass centers at their midpoints.

Reference frames:
- $R: N_1, N_2, N_3$ (fixed frame)
- $L_i : n_{i1}^1, n_{i2}^1, n_{i3}^1 \ (i = 1, 2)$ (fixed in the two links)

Find:
- a) $H_{G_i} \ (i = 1, 2)$ the angular momenta of the two bars about their respective mass centers
- b) $K_i \ (i = 1, 2)$ the kinetic energies of the bars

Solution:
- a) The inertia matrices of the links about the link-fixed directions can be written as
  \[
  \begin{bmatrix}
  I_{G_i} \end{bmatrix}_{L_i} = \frac{1}{12} m_i \ell_i^2 \\
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  \ (i = 1, 2)
  \]

  In Volume I, Exercise 5.2 it was found that for a 3-1-3 body fixed orientation angle sequence, the angular velocity vectors of the links can be written as follows.

  \[
  \mathbf{\omega L_i} = \mathbf{\omega n_{i1}^1 + \omega n_{i2}^2 + \omega n_{i3}^3 = (\dot{\theta}_{i1} S_{i2} S_{i3} + \dot{\theta}_{i2} C_{i3} ) n_{i1}^1 + (\dot{\theta}_{i1} S_{i2} C_{i3} - \dot{\theta}_{i2} S_{i3} ) n_{i2}^2 + (\dot{\theta}_{i3} + \dot{\theta}_{i1} C_{i2}) n_{i3}^3}
  \]

  The body-fixed components of the angular momenta of the links can then written as

  \[
  \begin{bmatrix}
  H_{G_i} \mathbf{\cdot n_{i1}^1} \\
  H_{G_i} \mathbf{\cdot n_{i2}^2} \\
  H_{G_i} \mathbf{\cdot n_{i3}^3}
  \end{bmatrix} = \frac{1}{12} m_i \ell_i^2 \\
  \begin{bmatrix}
  0 & 0 \\
  0 & 0 \\
  1 & 0
  \end{bmatrix} \mathbf{\cdot \omega L_i} = \frac{1}{12} m_i \ell_i^2 \\
  \begin{bmatrix}
  \omega_{i1} \\
  \omega_{i2} \\
  \omega_{i3}
  \end{bmatrix} = \frac{1}{12} m_i \ell_i^2 \\
  \begin{bmatrix}
  \omega_{i1} \\
  \omega_{i2} \\
  \omega_{i3}
  \end{bmatrix} \Rightarrow \mathbf{H_{G_i} = \frac{1}{12} m_i \ell_i^2 (\omega_{i1} n_{i1}^1 + \omega_{i3} n_{i3}^3)}
  \]

  b) The kinetic energies of the bars must include the translational and rotational energies. However, since link $OA$ is rotating about a fixed point, the kinetic energy could also be written as purely rotational energy about point $O$. 

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Link 1:
Taking advantage of the fact that link 1 is rotating about a fixed point, first find the angular momentum about the fixed point. Using the parallel axes theorem (or inertia tables directly) to find the inertias about the end of the link gives

\[
\begin{align*}
\vec{H}_o \cdot \vec{n}_1 &= \frac{1}{3} m_1 \ell_1^2 \\
\vec{H}_o \cdot \vec{n}_2 &= 0 \\
\vec{H}_o \cdot \vec{n}_3 &= \frac{1}{3} m_1 \ell_1^2 \\
\end{align*}
\]

Then the kinetic energy can be written as

\[
K_1 = \frac{1}{2} \omega_{l_1} \cdot \vec{H}_o = \frac{1}{2} \left( \frac{1}{3} m_1 \ell_1^2 \right) \left( \omega_{l_1}^2 + \omega_{l_3}^2 \right) = \frac{1}{6} m_1 \ell_1^2 \left( \omega_{l_1}^2 + \omega_{l_3}^2 \right)
\]

This result can also be found by using the more general form for the kinetic energy. Using this approach, first find the velocity of the mass center of the link.

\[
\begin{align*}
\frac{\dot{R}_{G_1}}{v_{G_1}} &= \frac{\dot{R}_{G_1}}{v_{G_1}} + R_{G_1} \times L_{G_1} \\
\frac{\dot{R}_{G_1}}{v_{G_1}} &= \frac{\dot{R}_{G_1}}{v_{G_1}} \times L_{G_1} = \left( \omega_{l_1} n_1^1 + \omega_{l_2} n_2^1 + \omega_{l_3} n_3^1 \right) \times \left( -\frac{\ell_1}{2} n_2^1 \right) = -\frac{\ell_1}{2} \left( \omega_{l_1} n_3^1 - \omega_{l_3} n_1^1 \right)
\end{align*}
\]

Using the general form for kinetic energy gives the same result

\[
K = \frac{1}{2} m_1 \left( \frac{\dot{R}_{G_1}}{v_{G_1}} \right)^2 + \frac{1}{2} \frac{\dot{R}_{G_1}}{v_{G_1}} \cdot \vec{H}_G
\]

\[
\Rightarrow K_1 = \frac{1}{6} m_1 \ell_1^2 \left( \omega_{l_1}^2 + \omega_{l_3}^2 \right)
\]

Link 2:
First find the velocity of the mass center of the link.

\[
\begin{align*}
\frac{\dot{R}_{G_2}}{v_{G_2}} &= \frac{\dot{R}_{G_2}}{v_{G_2}} + \frac{\dot{R}_{G_2}}{v_{G_2}} = \left( \frac{\dot{R}_{G_2}}{v_{G_2}} \times L_{G_2} \right) + \left( \frac{\dot{R}_{G_2}}{v_{G_2}} \times L_{G_2} \right) \\
\frac{\dot{R}_{G_2}}{v_{G_2}} &= \left( \omega_{l_1} n_1^1 + \omega_{l_2} n_2^1 + \omega_{l_3} n_3^1 \right) \times \left( -\ell_1 n_2^1 \right) + \left( \omega_{l_2} n_2^1 + \omega_{l_3} n_3^1 \right) \times \left( -\ell_2 n_2^1 \right) \\
\frac{\dot{R}_{G_2}}{v_{G_2}} &= -\ell_1 \left( \omega_{l_1} n_3^1 - \omega_{l_3} n_1^1 \right) - \frac{\ell_2}{2} \left( \omega_{l_2} n_3^1 - \omega_{l_3} n_1^1 \right) \\
\Rightarrow \frac{\dot{R}_{G_2}}{v_{G_2}} &= -\ell_1 \left( \omega_{l_1} n_3^1 - \omega_{l_3} n_1^1 \right) + \frac{\ell_2}{2} \left( \omega_{l_2} n_3^1 - \omega_{l_3} n_1^1 \right)
\end{align*}
\]

Transformation matrices can now be used to resolve all the components of \( \frac{\dot{R}_{G_2}}{v_{G_2}} \) into the base frame.
\[ R_{\mathcal{V}G_2} = \ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} n_1^T \\ n_2 \\ n_3 \end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} n_1^T \\ n_2 \\ n_3 \end{bmatrix} \]

\[ = \ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \]

So, the components of \( R_{\mathcal{V}G_2} \) in the base system are

\[ \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}^T = \left[ \ell_1 \begin{bmatrix} \omega_{13} & 0 & -\omega_{11} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} \omega_{23} & 0 & -\omega_{21} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \right]^T \]

\[ \Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \ell_1 \begin{bmatrix} N_1 \end{bmatrix}^T \begin{bmatrix} \omega_{13} \\ 0 \\ -\omega_{11} \end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} N_1 \end{bmatrix}^T \begin{bmatrix} \omega_{23} \\ 0 \\ -\omega_{21} \end{bmatrix} \]

The kinetic energy of link 2 can now be written as

\[ K_2 = \frac{1}{2} m_2 \left( R_{\mathcal{V}G_2} \right)^2 + \frac{1}{2} R_{\mathcal{V}G_2} \cdot H_{G_2} \]

\[ = \frac{1}{2} m_2 \left( V_1^2 + V_2^2 + V_3^2 \right) + \frac{1}{2} \left( \omega_{21} n_1^2 + \omega_{22} n_2^2 + \omega_{23} n_3^2 \right) \left( \frac{1}{12} m_2 \ell_2^2 \left( \omega_{21} n_1^2 + \omega_{23} n_3^2 \right) \right) \]

\[ \Rightarrow K_2 = \frac{1}{2} m_2 \left( V_1^2 + V_2^2 + V_3^2 \right) + \frac{1}{2} m_2 \ell_2^2 \left( \omega_{21}^2 + \omega_{23}^2 \right) \]

Clearly, the most intricate part of the kinetic energy of link 2 is in the translational energy.
Exercises

The following observations are helpful in the solution of problems 1.1 and 1.2.

1. If \( \alpha \) is an arbitrary scalar, and if \( \lambda \) is an eigenvalue of matrix \([A]\), then \( \alpha \lambda \) is an eigenvalue of matrix \( \alpha [A] \).

2. If \( \alpha \) is an arbitrary scalar, and if \( \vec{x} \) is an eigenvector of matrix \([A]\), then \( \alpha \vec{x} \) is also an eigenvector of matrix \( \alpha [A] \) corresponding to the eigenvalue \( \alpha \lambda \).

1.1 The body shown consists of two L-shaped arms welded to a straight rod. The straight segment has length \( 3a \), and each segment of the L-shaped arms has length \( a \). Each segment of length \( a \) has mass \( m \). All segments are slender.

a) Find the principal moments of inertia and the principal directions for the mass-center \( G \).

b) Show that the eigenvector (or modal) matrix found in part (a) diagonalizes the inertia matrix.

Answers:

\[
[I_G] = ma^2 \begin{bmatrix} \frac{47}{12} & 1 & -\frac{7}{2} \\ 1 & \frac{71}{12} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{10}{3} \end{bmatrix} \approx ma^2 \begin{bmatrix} 3.91667 & 1.00000 & -1.50000 \\ 1.00000 & 5.91667 & 0.500000 \\ -1.50000 & 0.500000 & 3.33333 \end{bmatrix}
\]

\[
[M] \approx \begin{bmatrix} 0.642082 & -0.648393 & -0.409045 \\ -0.246360 & 0.330748 & -0.910995 \\ 0.725974 & 0.685706 & 0.052629 \end{bmatrix}
\]

\[\det[M] = 1\]

\[I_1 \approx 1.83699 \, m \, a^2 \quad I_2 \approx 4.99288 \, m \, a^2 \quad I_3 \approx 6.33679 \, m \, a^2\]

\[\varepsilon_1 = 0.642082 \, N_1 - 0.246360 \, N_2 + 0.725974 \, N_3\]

\[\varepsilon_2 = -0.648393 \, N_1 + 0.330748 \, N_2 + 0.685706 \, N_3\]

\[\varepsilon_3 = -0.409045 \, N_1 - 0.910995 \, N_2 + 0.052629 \, N_3\]

1.2 The figures below show two views of a body with a central cylindrical section and two identical, box-like ends. The central cylindrical section has a diameter of \( \ell \) and length of 10\( \ell \). The box-like ends have two square sides (length and width equal to 4\( \ell \)) and a depth of \( \ell \). The cylinder has mass \( m \) and the box-like ends each have mass \( 2m \), so the total mass of the composite shape is \( 5m \). Find the principal moments of inertia and the principal directions for the point \( O \) on the outer corner of end \( B \).
1.3 The rectangular plate $P$ is welded to a shaft so that it rotates about its diagonal. (a) Find $\mathbf{H}_G$ the angular momentum of $P$ about its mass center $G$. Express your results in the $X, Y, \text{ and } Z' \text{ directions. (b) Find } K \text{ the kinetic energy of the plate.}$

**Answers:**

\[
[I_o] = ml^2 \begin{bmatrix}
\frac{1210}{24} & -60 & -60 \\
-60 & \frac{5361}{16} & -20 \\
-60 & -20 & \frac{5361}{16}
\end{bmatrix} \approx ml^2 \begin{bmatrix}
50.792 & -60 & -60 \\
-60 & 335.06 & -20 \\
-60 & -20 & 335.06
\end{bmatrix}
\]

\[
[M] \approx \begin{bmatrix}
0.959542 & -0.281564 & 0.000000 \\
0.199096 & 0.678499 & -0.707107 \\
0.199096 & 0.678499 & 0.707107
\end{bmatrix}
\]

\[
det [M] = 1
\]

\[
I_1 \approx 25.8928 ml^2 \quad I_2 \approx 339.961 ml^2 \quad I_3 = 355.062 ml^2
\]

\[
\epsilon_1 = 0.959542 N_1 + 0.199096 N_2 + 0.199096 N_3
\]

\[
\epsilon_2 = -0.281564 N_1 + 0.678499 N_2 + 0.678499 N_3
\]

\[
\epsilon_3 = -0.707107 N_2 + 0.707107 N_3
\]

1.4 The system shown consists of two L-shaped arms welded to a shaft of length $3a$. The planes of the arms are at right angles to the shaft. If all parts are made of “slender” bars, complete the following. (a) Find $\mathbf{H}_G$ the angular momentum of the system about its mass center $G$. Express your results in the $X', Y', \text{ and } Z' \text{ directions. (b) Find } K \text{ the kinetic energy of the system.}$

**Answers:**

a) $\mathbf{H}_G = \frac{ma^2 b \omega}{12(a^2 + b^2)} \left( 2ab i + (a^2 - b^2) j' \right)$  

b) $K = ma^2 b^2 \omega^2 / 12(a^2 + b^2)$
1.5 The system shown consists of a bar $B$ that is pinned through the center of a shaft of length $2a$. As the shaft rotates about the $Z$-axis at a rate $\Omega$ (r/s), $B$ rotates about the $Y'$ at a rate $\dot{\theta} = \omega$ (r/s). (a) Find $\mathbf{H}_G$ the angular momentum of $B$ about its mass center $G$. Express your results in the $X'$, $Y'$, and $Z$ directions. (b) Find $K$ the kinetic energy of $B$.

Answers:

a) $\mathbf{H}_G = \frac{1}{12} mL^2 \left( - (\Omega \dot{S}_\phi C_\phi) \mathbf{i}' + \omega \mathbf{j}' + (\Omega^2 S_\phi^2) \mathbf{k} \right)$

b) $K = \frac{1}{2} m L^2 \left( \omega^2 + \Omega^2 S_\phi^2 \right)$

1.6 The system shown consists of a bar $B$ that is pinned to the bottom of a disk $D$. As the disk rotates at a rate $\Omega$ (rad/sec) about the $Z$-axis, the bar rotates at a rate $\dot{\theta}$ (rad/sec) about the $X'$ direction. (a) Find $\mathbf{H}_G$ the angular momentum of $B$ about its mass center $G$. Express your results in the $X'$, $Y'$, and $Z$ directions. (b) Find $K$ the kinetic energy of $B$.

Answers:

a) $\mathbf{H}_G = \frac{1}{12} mL^2 \left( \dot{\theta} \mathbf{i}' + (\Omega S_\phi C_\phi) \mathbf{j}' + (\Omega^2 S_\phi^2) \mathbf{k} \right)$

b) $K = \frac{1}{2} m \left( B \Omega + \frac{1}{2} (\Omega S_\phi) \right)^2 + \frac{1}{2} mL^2 \dot{\theta}^2 + \frac{1}{2} mL^2 \Omega^2 S_\phi^2$

1.7 The system shown consists of two bodies, the cross-shaped frame $A$ and the disk $D$. Frame $A$ is connected to the ground with a two-axis joint whose motion is described by the angles $\phi$ and $\theta$. The angle $\phi$ allows $A$ to rotate about a vertical axis while the angle $\theta$ allows an additional rotation about the rotating $n_2$ direction. Disk $D$ is pinned to the end of $A$ and can rotate relative to $A$ also about the $n_2$ direction. The unit vector set $\mathbf{A} : (n_1, n_2, n_3)$ are fixed in the frame $A$.

The points $G_A$ and $G_D$ represent the mass centers of $A$ and $D$. a) Find $\mathbf{H}_{G_A}$ and $\mathbf{H}_{G_D}$ the angular momenta of $A$ and $D$ about their mass centers, and b) Find $K$ the kinetic energy of the system. The system mass center $G$ lies a distance $d_A$ to the right of $G_A$ and a distance $d_D$ to the left of $G_D$.

Answers:

a) $\mathbf{H}_{G_A} = I_{11}^{G_A} \omega_1 n_1 + I_{21}^{G_A} \omega_2 n_2 + I_{31}^{G_A} \omega_3 n_3$

b) $\mathbf{H}_{G_D} = I_{11}^{G_D} \omega_1 n_1 + I_{22}^{G_D} \omega_2 n_2 + I_{33}^{G_D} \omega_3 n_3$
\[ K = \frac{1}{2} \left( m_A + 4m_D \right) b^2 \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right) + \frac{1}{2} \left[ I_{11}^{G_0} \omega_1^2 + I_{22}^{G_0} \left( \omega_2 + \omega_D \right)^2 + I_{33}^{G_0} \omega_3^2 \right] \]

or

\[ K = \frac{1}{2} m_T \left( b + d_A \right)^2 \left( \omega_1^2 + \omega_2^2 + \omega_3^2 \right) + \frac{1}{2} \left[ \left( I_{11}^{G} \right)_{A+D} \omega_1^2 + \left( I_{22}^{G} \right)_{A+D} \omega_2^2 + \left( I_{33}^{G} \right)_{A+D} \omega_3^2 \right] \]

\[ + \frac{1}{2} \left( I_{22}^{G} \right)_D \left( \omega_D^2 + 2\omega_2\omega_D \right) \]

1.8 The system shown is a three-dimensional double pendulum or arm. The first link is connected to ground and the second link is connected to the first with universal joints at O and A, respectively. The ground frame is \( R : (N_1, N_2, N_3) \) and the link frames are \( L_i : (n_{i1}, n_{i2}, n_{i3}) \) \((i = 1, 2)\). The orientation of \( L_1 \) is defined relative to \( R \) and the orientation of \( L_2 \) is defined relative to \( L_1 \) each with a 1-3 body-fixed rotation sequence.

Link \( OA \) is oriented relative to the ground frame by first rotating through an angle \( \theta_{11} \) about the \( N_1 \) direction, and then rotating about an angle \( \theta_{12} \) about the \( n_{13}^1 \) direction. Link \( AB \) is oriented relative to link \( OA \) by rotating first through an angle \( \theta_{21} \) about the \( n_{13}^1 \) direction, and then through an angle \( \theta_{22} \) about the \( n_{23}^2 \) direction. The lengths of the links are \( \ell_1 \) and \( \ell_2 \) with mass centers are at their midpoints. Find a) \( H_{G_i} \) \((i = 1, 2)\) the angular momenta of the two bars about their respective mass centers, and b) \( K_i \) \((i = 1, 2)\) the kinetic energies of the bars.

Answers:

\[ \mathbf{K}_{L_1} = \frac{1}{6} m_1 \ell_{11}^2 \left( \omega_{11}^2 + \omega_{13}^2 \right) \]

\[ \mathbf{K}_{L_2} = \frac{1}{2} m_2 \left( V_1^2 + V_2^2 + V_3^2 \right) + \frac{1}{24} m_2 \ell_{21}^2 \left( \omega_{21}^2 + \omega_{23}^2 \right) \]
\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix}
\omega_{13} \\
0 \\
-\omega_{11}
\end{bmatrix} + \frac{\ell_2}{2} \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix}
\omega_{23} \\
0 \\
-\omega_{21}
\end{bmatrix}
\]

\[
R_i = \begin{bmatrix}
C_{i2} & C_{i1}S_{i2} & S_{i1}S_{i2} \\
-S_{i2} & C_{i1} & S_{i1}C_{i2} \\
0 & -S_{i1} & C_{i1}
\end{bmatrix}
\]

References:


Addendum on Inertia – Nondistinct Eigenvalues

It was noted above that nonsymmetric bodies can have an infinite number of inertia matrices associated with each point in the body, because the inertia matrix changes with the orientation of the axes at that point. All those inertia matrices can generally be reduced to a single unique inertia matrix about a unique set of principal axes at that point. However, for this to be true, the principal moments of inertia must be distinct.

Symmetric bodies also have a unique set of principal moments of inertia at any point. However, symmetric bodies can have multiple sets of principal axes at a given point and multiple points can have the same principal moments of inertia and principal axes. A body of revolution was used above as an example.

It turns out that a point of a body has multiple sets of principal axes whenever all the eigenvalues of the inertia matrix for that point are not distinct, that is whenever two or all three of the eigenvalues are equal. The body may be symmetrical about the plane formed by the eigenvectors, but it may not.
Example 6: Square Prism

To illustrate this situation, consider the square prism shown below having square ends with sides of length $a$ and a total prism length of $3a$. The mass center $G$ is at the center of the prism and the axes of the reference frame $B: (x, y, z)$ located at $G$ are perpendicular to and pass through the centers of the sides as shown.

The $x$-$y$, $x$-$z$, and $y$-$z$ planes are all planes of symmetry, so the three axes are principal axes and the moments of inertia about these axes are the principal moments of inertia. Using a set of standard tables, the principal inertias and inertia matrix can be written as follows.

$$I_{xx} = \frac{1}{12}m(a^2 + a^2) = \frac{1}{6}ma^2$$
$$I_{yy} = I_{zz} = \frac{1}{12}m(a^2 + (3a)^2) = \frac{10}{12}ma^2 = \frac{5}{6}ma^2$$

$$\begin{bmatrix} I_G^b \end{bmatrix} = \frac{ma^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now consider rotating frame $B$ relative to a frame $C: (X, Y, Z)$ using a 1-2-3 body-fixed rotation sequence as defined in Unit 5 of Volume I. The transformation matrix that transforms vector components from $C$ into $B$ can be written as follows.

$$[R] = [R_3][R_2][R_1] = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 & S_1 \\ -S_1 & C_1 \end{bmatrix} = \begin{bmatrix} C_2C_3 & C_1S_3 + S_3C_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1C_2S_3 \\ C_3 & S_3C_1 & C_1C_2 \end{bmatrix}$$

Here, $S_i$ ($i = 1, 2, 3$) and $C_i$ ($i = 1, 2, 3$) represent the sines and cosines of orientation angles $\theta_1$, $\theta_2$, and $\theta_3$. Using results presented earlier in this unit, the inertia matrix about the axes of frame $C$ can be calculated as follows.

$$\begin{bmatrix} I_G^c \end{bmatrix} = [R]^T \begin{bmatrix} I_G^b \end{bmatrix} [R]$$

As an example, consider rotating frame $B$ relative to $C$ using the angles $\theta_1 = 20$ (deg), $\theta_2 = -30$ (deg), and $\theta_3 = 60$ (deg). Using these values, the transformation matrix $[R]$ is found to approximately be
Using this transformation matrix, the inertia matrix about the axes of reference frame \( C \) is found to be

\[
\begin{bmatrix}
I^C_G
\end{bmatrix}
= \begin{bmatrix}
[R]^T
\end{bmatrix}
\begin{bmatrix}
I^B_G
\end{bmatrix}
\begin{bmatrix}
[R]
\end{bmatrix}
\approx \frac{ma^2}{6}
\begin{bmatrix}
4.25 & -1.26144 & -0.91993 \\
-1.26144 & 2.87836 & -1.54725 \\
-0.91993 & -1.54725 & 3.87164
\end{bmatrix}
\]

In theory, this process can now be reversed by calculating the eigenvalues and eigenvectors of \( \begin{bmatrix}
I^C_G
\end{bmatrix} \).

As noted in the Exercises, if \( \alpha \) is an arbitrary scalar, and if \( \lambda \) is an eigenvalue of matrix \( \begin{bmatrix}
A
\end{bmatrix} \), then \( \alpha \lambda \) is an eigenvalue of \( \alpha \begin{bmatrix}
A
\end{bmatrix} \). Also, if \( \vec{v} \) is an eigenvector of matrix \( \begin{bmatrix}
A
\end{bmatrix} \), the \( \vec{v} \) is also an eigenvector of matrix \( \alpha \begin{bmatrix}
A
\end{bmatrix} \). So, if we define the matrix \( \begin{bmatrix}
A
\end{bmatrix} \) as

\[
\begin{bmatrix}
4.25 & -1.26144 & -0.91993 \\
-1.26144 & 2.87836 & -1.54725 \\
-0.91993 & -1.54725 & 3.87164
\end{bmatrix}
\]

then the eigenvalues of \( \begin{bmatrix}
I^C_G
\end{bmatrix} \) are \( \frac{ma}{6} \) times the eigenvalues of \( \begin{bmatrix}
A
\end{bmatrix} \). Also, the eigenvectors of \( \begin{bmatrix}
I^C_G
\end{bmatrix} \) are the same as the eigenvectors of \( \begin{bmatrix}
A
\end{bmatrix} \).

Using MATLAB, the eigenvalues and eigenvectors of \( \begin{bmatrix}
I^C_G
\end{bmatrix} \) are found to approximately be

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\approx \frac{ma^2}{6}
\begin{bmatrix}
1 \\
5 \\
5
\end{bmatrix}
\]

\[
\begin{bmatrix}
M
\end{bmatrix}_B
\approx
\begin{bmatrix}
1 & 0 & 0 \\
0 & -0.960465 & -0.278402 \\
0 & -0.278402 & 0.960465
\end{bmatrix}
\]

\[
\begin{bmatrix}
M
\end{bmatrix}_C
\approx
\begin{bmatrix}
0.43301 & 0.85955 & -0.27143 \\
0.72829 & -0.51105 & -0.45653 \\
0.53112 & 0 & 0.84730
\end{bmatrix}
\]

Here, the components of the eigenvectors in frame \( B \) form the columns of the matrix \( \begin{bmatrix}
M
\end{bmatrix}_B \) and the components of the eigenvectors in frame \( C \) form the columns of the matrix \( \begin{bmatrix}
M
\end{bmatrix}_C \). Notice the first eigenvector (first column of \( \begin{bmatrix}
M
\end{bmatrix}_B \)) points along the \( x \) axis as we expect. However, the second and third do not point along the \( y \) and \( z \) axes. In fact, the second and third eigenvectors are perpendicular to each other but have both \( y \) and \( z \) components. The second eigenvector makes an angle \( \phi \approx 73.8 \) (deg) with the \( z \) axis. See the pair of blue vectors in the diagram.
Are these principal axes? Certainly, none of the planes associated with the x axis and these two eigenvectors are planes of symmetry. To answer this question, consider rotating frame $B$ relative to frame $C$ using a single rotation through an angle $\theta$ about the x axis. In this case, the transformation matrix that transforms vector components from $C$ into $B$ is

$$
[R] = [R_1] = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & C_i & S_i \\
0 & -S_i & C_i
\end{bmatrix}
$$

The inertia matrix about the axes of frame $C$ can then be calculated as follows.

$$
[I^c_G] = [R_1]^T [I^B_G] [R_1] = \frac{ma^2}{6} \begin{bmatrix}
1 & 0 & 0 \\
0 & C_i & -S_i \\
0 & S_i & C_i
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & C_i & S_i \\
0 & -S_i & C_i
\end{bmatrix} = \frac{ma^2}{6} \begin{bmatrix}
1 & 0 & 0 \\
0 & 5(C_i^2 + S_i^2) & 5S_iC_i \\
0 & 5C_iS_i & 5(S_i^2 + C_i^2)
\end{bmatrix}
$$

$$
\Rightarrow [I^c_G] = \frac{ma^2}{6} \begin{bmatrix}
1 & 0 \\
0 & 5 \\
0 & 0 & 5
\end{bmatrix} = [I^B_G]
$$

So, the inertia matrix remains unchanged by an arbitrary rotation about the x axis.

As a final step in this example, we can make sure the eigenvector matrix can be used to diagonalize the inertia matrix. For this we need the eigenvectors expressed in the frame $C$.

$$
[I^c_G][M]_C \approx \frac{ma^2}{6} \begin{bmatrix}
4.25 & -1.26144 & -0.91993 \\
-1.26144 & 2.87836 & -1.54725 \\
-0.91993 & -1.54725 & 3.87164
\end{bmatrix} \begin{bmatrix}
0.43301 & 0.85955 & -0.27143 \\
0.72829 & -0.51105 & -0.45653 \\
0.53112 & 0 & 0.84730
\end{bmatrix}
$$

$$
\approx \frac{ma^2}{6} \begin{bmatrix}
0.43301 & 4.29775 & -1.35715 \\
0.72829 & -2.55526 & -2.28263 \\
0.53112 & 0 & 4.23648
\end{bmatrix}
$$
\[
\begin{bmatrix}
0.43301 & 0.72829 & 0.53112 \\
0.85955 & -0.51105 & 0 \\
-0.27143 & -0.45653 & 0.84730
\end{bmatrix}
\approx \frac{ma^2}{6}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{bmatrix}
\]

So, in fact, the eigenvector matrix can be used to diagonalize the inertia matrix.

Final Note

Although the above result was derived specifically for the square prism, the results are true so long as the inertias about two of the principal axes are equal. Hence, for a body, if two of the principal inertias are equal, then – 1) any axis perpendicular to the axis associated with the third (distinct) principal inertia is a principal axis, and 2) an arbitrary rotation about the axis associated with the third (distinct) principal inertia does not change the inertia matrix. This result can be extended to show that all vectors are eigenvectors if all three principal inertias are equal.