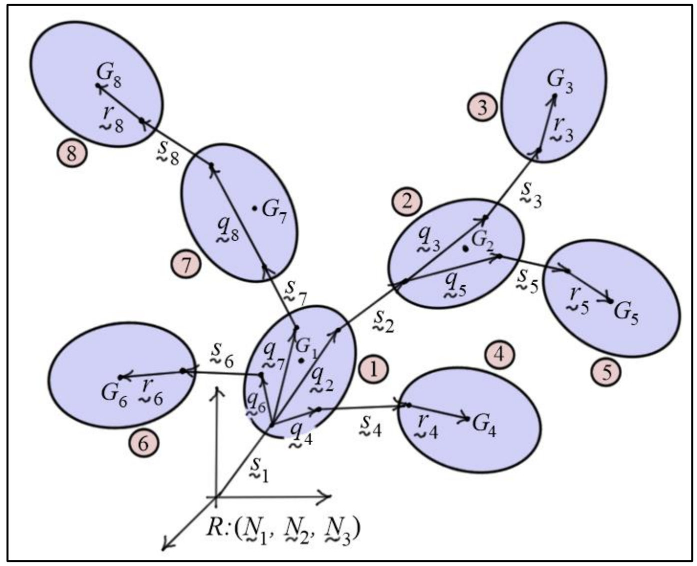
Multibody Dynamics
Equations of Motion for Unconstrained Systems Using Relative Coordinates with Euler Parameters

As mentioned in previous notes, the ***explicit form*** of the equations of motion of a multibody system depends on:

- choice of generalized coordinates
- choice of generalized speeds
- method used to formulate equations
- constraints on system motion

In these notes, ***Kane's equations*** are used to derive the equations of motion of a multibody system using ***relative coordinates*** to describe the ***relative orientation*** and ***relative translation*** between adjoining bodies.



Multibody System with Eight Bodies

Using relative coordinates, the ***kinematic analysis*** is generally ***more complicated***, but the ***constraints*** between adjoining bodies are usually more ***straight-forward*** to formulate. ***Recursive relationships*** can be developed for kinematic variables to streamline the analysis.

In the analysis that follows, the ***relative orientations*** and ***angular velocities*** of bodies are described using ***Euler parameters*** and ***angular velocity components***. As discussed in previous notes, a ***body-connection array*** is used as an aid when developing the kinematic and dynamic equations of motion. For convenience (and without loss of generality), it is assumed herein that the bodies are numbered starting with "1" as the ***reference body*** and ***increasing*** the ***body numbers*** while moving outward along the branches, so a body's ***lower-body*** is also a ***lower-numbered body***.

<u>Generalized Coordinates and Speeds</u>

The generalized coordinates and generalized speeds for a system with "N" bodies are listed below.

- ***Euler parameters*** $\hat{\varepsilon}_{Ki}$ $(K=1,\dots,N;\ i=1,2,3,4)$ are used to measure the ***orientations*** of the bodies ***relative*** to their ***adjacent, lower bodies***. So, the Euler parameters $\hat{\varepsilon}_{Ki}$ $(i=1,2,3,4)$ measure the orientation of body K relative to body $\mathscr{L}(K)$.
- ***Translation variables*** s'_{Ki} $(K=1,\dots,N;\ i=1,2,3)$ are used to measure ***displacements*** of the bodies ***relative*** to their ***adjacent, lower bodies***. These variables represent the ***lower-body-frame components*** of the translation vectors of the bodies $(\underset{\sim}{s}_K)$.
- ***Relative angular velocity components*** $\hat{\omega}'_{Ki}$ $(K=1,\dots,N;\ i=1,2,3)$ are used to measure the angular velocities of the bodies relative to their ***adjacent, lower bodies***. These are the ***body-frame components*** of the ***relative angular velocity*** vectors of the bodies $\left(\hat{\underset{\sim}{\omega}}_K \triangleq {}^{\mathscr{L}(K)}\underset{\sim}{\omega}_K\right)$.

As described, there are "$7N$" ***generalized coordinates***, $\hat{\varepsilon}_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3,4)$ and s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$. To avoid the use of Lagrange multipliers, the "$6N$" ***generalized speeds*** are defined to be $\hat{\omega}'_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$ and $\dot{s}'_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$.

<u>System State Vectors</u>

Using the ***generalized coordinates*** and ***speeds*** defined above, the following ***system state vectors*** can be defined.

$$\{\hat{\varepsilon}\}_{4N\times 1}=[\hat{\varepsilon}_{11},\hat{\varepsilon}_{12},\hat{\varepsilon}_{13},\hat{\varepsilon}_{14},\ldots,\underbrace{\hat{\varepsilon}_{K1},\hat{\varepsilon}_{K2},\hat{\varepsilon}_{K3},\hat{\varepsilon}_{K4}}_{\{\hat{\varepsilon}_K\}^T},\ldots,\hat{\varepsilon}_{N1},\hat{\varepsilon}_{N2},\hat{\varepsilon}_{N3},\hat{\varepsilon}_{N4}]^T$$

$$\{s'\}_{3N\times 1}=[s'_{11},s'_{12},s'_{13},\ldots,\underbrace{s'_{K1},s'_{K2},s'_{K3}}_{\{s'_K\}^T},\ldots,s'_{N1},s'_{N2},s'_{N3}]^T$$

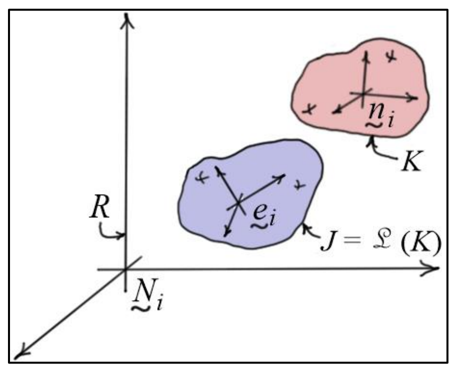
$$\{\hat{\omega}'\}_{3N\times 1}=[\hat{\omega}'_{11},\hat{\omega}'_{12},\hat{\omega}'_{13},\ldots,\underbrace{\hat{\omega}'_{K1},\hat{\omega}'_{K2},\hat{\omega}'_{K3}}_{\{\hat{\omega}'_K\}^T},\ldots,\hat{\omega}'_{N1},\hat{\omega}'_{N2},\hat{\omega}'_{N3}]^T$$

and

$$\{x\}_{7N\times 1}=\begin{Bmatrix}\{x_1\}\\ \{x_2\}\end{Bmatrix}=\begin{Bmatrix}\{\hat{\varepsilon}\}_{4N\times 1}\\ \{s'\}_{3N\times 1}\end{Bmatrix}\qquad\qquad \{y\}_{6N\times 1}=\begin{Bmatrix}\{y_1\}\\ \{y_2\}\end{Bmatrix}=\begin{Bmatrix}\{\hat{\omega}'\}_{3N\times 1}\\ \{\dot{s}'\}_{3N\times 1}\end{Bmatrix}\qquad (1)$$

<u>Transformation Matrices</u>

Consider two bodies of the multibody system. Body J is the adjacent, lower body of body K, that is, $J=\mathscr{L}(K)$. The unit vectors of the two bodies can be written in terms of the inertial frame vectors using the body transformation matrices.



$$\boxed{\{\underset{\sim}{e}\}=\left[R_J\right]\{\underset{\sim}{N}\}}\qquad\qquad \boxed{\{\underset{\sim}{n}\}=\left[R_K\right]\{\underset{\sim}{N}\}}$$

Using these two results, the unit vectors in body K can be written in terms of the unit vectors of body J as follows

$$\boxed{\{\underset{\sim}{n}\}=\left[R_K\right]\{\underset{\sim}{N}\}=\left[R_K\right]\left[R_J\right]^T\{\underset{\sim}{e}\}\triangleq\left[{}^JR_K\right]\{\underset{\sim}{e}\}}\qquad (2)$$

where $\left[{}^JR_K\right]$ is the relative transformation matrix used to write the unit vectors of body K in terms of the unit vectors of body J.

$$\boxed{\left[{}^JR_K\right]\triangleq\left[R_K\right]\left[R_J\right]^T}$$

Or, ***multiplying*** both sides of the equation on the ***right*** by $\left[R_J\right]$ gives

$$\boxed{\left[R_K\right]=\left[{}^JR_K\right]\left[R_J\right]}\qquad (3)$$

The relative transformation matrix $\left[{}^{J}R_K\right]$ can be written in terms of the Euler parameters as follows.

$$\left[{}^{J}R_K\right]=\begin{bmatrix} (\hat{\varepsilon}_{K1}^2-\hat{\varepsilon}_{K2}^2-\hat{\varepsilon}_{K3}^2+\hat{\varepsilon}_{K4}^2) & 2(\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K2}+\hat{\varepsilon}_{K3}\hat{\varepsilon}_{K4}) & 2(\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K3}-\hat{\varepsilon}_{K2}\hat{\varepsilon}_{K4}) \\ 2(\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K2}-\hat{\varepsilon}_{K3}\hat{\varepsilon}_{K4}) & (-\hat{\varepsilon}_{K1}^2+\hat{\varepsilon}_{K2}^2-\hat{\varepsilon}_{K3}^2+\hat{\varepsilon}_{K4}^2) & 2(\hat{\varepsilon}_{K2}\hat{\varepsilon}_{K3}+\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K4}) \\ 2(\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K3}+\hat{\varepsilon}_{K2}\hat{\varepsilon}_{K4}) & 2(\hat{\varepsilon}_{K2}\hat{\varepsilon}_{K3}-\hat{\varepsilon}_{K1}\hat{\varepsilon}_{K4}) & (-\hat{\varepsilon}_{K1}^2-\hat{\varepsilon}_{K2}^2+\hat{\varepsilon}_{K3}^2+\hat{\varepsilon}_{K4}^2) \end{bmatrix} \tag{4}$$

The result in Eq. (3) is ***easily extended*** to include as ***many bodies*** as necessary to move from a ***body-frame*** to a ***fixed-frame through*** frames of a ***series*** of ***interconnected bodies***.

$$\left[R_K\right]=\left[{}^{\mathscr{L}(K)}R_K\right]\left[{}^{\mathscr{L}^2(K)}R_{\mathscr{L}(K)}\right]\cdots\left[{}^{\mathscr{L}^{u_K}(K)}R_{\mathscr{L}^{u_K-1}(K)}\right]\left[R_{\mathscr{L}^{u_K}(K)}\right] \tag{5}$$

Recall that $\mathscr{L}^{u_K}(K)=1$ refers to the ***reference body*** of the system.

Time-Derivatives of the Transformation Matrices

In previous notes, the ***time derivatives*** of the ***transformation matrices*** from the bodies to the ***inertial frame*** were written in terms of ***skew-symmetric matrices*** formed from the components of the ***angular velocities*** of the bodies ***relative*** to the ***fixed frame***. Recall that the "prime" indicates the use of ***body-frame components***.

$$\left[\dot{R}_K\right]=\left[\tilde{\omega}'_K\right]^T\left[R_K\right] \quad \text{(components of } {}^{R}\underset{\sim}{\omega}_K \text{ are resolved in body } K\text{)} \tag{6}$$

The ***time derivatives*** of the ***transformation matrices between bodies*** in the system were written in terms of ***skew-symmetric matrices*** formed from components of the ***angular velocities*** of the bodies ***relative*** to their ***adjacent, lower bodies***. In the formula that follows, body *J* is the ***lower body*** of body *K*, and the "prime" indicates components resolved in body *K*.

$$\left[{}^{J}\dot{R}_K\right]=\left[{}^{J}\tilde{\omega}'_K\right]^T\left[{}^{J}R_K\right] \quad \text{(components of } {}^{J}\underset{\sim}{\omega}_K \text{ are resolved in body } K\text{)} \tag{7}$$

Relative Angular Velocity Components and Euler Parameters

The ***relative angular velocity components*** of body *K* can be written in terms of its Euler parameters as follows

$$\begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \\ 0 \end{Bmatrix}=2\begin{bmatrix} \hat{\varepsilon}_{K4} & \hat{\varepsilon}_{K3} & -\hat{\varepsilon}_{K2} & -\hat{\varepsilon}_{K1} \\ -\hat{\varepsilon}_{K3} & \hat{\varepsilon}_{K4} & \hat{\varepsilon}_{K1} & -\hat{\varepsilon}_{K2} \\ \hat{\varepsilon}_{K2} & -\hat{\varepsilon}_{K1} & \hat{\varepsilon}_{K4} & -\hat{\varepsilon}_{K3} \\ \hat{\varepsilon}_{K1} & \hat{\varepsilon}_{K2} & \hat{\varepsilon}_{K3} & \hat{\varepsilon}_{K4} \end{bmatrix}\begin{Bmatrix} \dot{\hat{\varepsilon}}_{K1} \\ \dot{\hat{\varepsilon}}_{K2} \\ \dot{\hat{\varepsilon}}_{K3} \\ \dot{\hat{\varepsilon}}_{K4} \end{Bmatrix}\triangleq 2\left[\hat{E}'_K\right]\left\{\dot{\hat{\varepsilon}}_K\right\} \tag{8}$$

Note the last equation is simply the derivative of the Euler parameter constraint equation, $\hat{\varepsilon}_1^2+\hat{\varepsilon}_2^2+\hat{\varepsilon}_3^2+\hat{\varepsilon}_4^2=1$.

The matrix $\left[\hat{E}'_K\right]$ is an ***orthogonal matrix***, so the above equation can be easily ***inverted*** to give

$$\left\{\dot{\hat{\varepsilon}}_K\right\}=\tfrac{1}{2}\left[\hat{E}'_K\right]^T\left\{\hat{\omega}'_K\right\}_{4\times1} \tag{9}$$

The subscript "4×1" has been ***added*** as a reminder that ***zero*** has been added as the ***fourth element*** of the angular velocity column vector. In practice, the fourth column of $\left[\hat{E}'_K\right]^T$ and the fourth row of $\{\hat{\omega}'_K\}_{4\times1}$ can be ***eliminated*** (because the fourth element of $\{\hat{\omega}'_K\}_{4\times1}$ is zero).

Angular Velocity and Partial Angular Velocity

The angular velocity of body K can be found using the summation rule for angular velocities and the body-connection array.

$$ {}^R\underset{\sim}{\omega}_K = \hat{\underset{\sim}{\omega}}_1 + \hat{\underset{\sim}{\omega}}_{\mathscr{L}^{u_K-1}(K)} + \cdots + \hat{\underset{\sim}{\omega}}_{\mathscr{L}(K)} + \hat{\underset{\sim}{\omega}}_K = \sum_{i=u_K}^{0} \hat{\underset{\sim}{\omega}}_{\mathscr{L}^i(K)} = \sum_{i=u_K}^{1} \hat{\underset{\sim}{\omega}}_{\mathscr{L}^i(K)} + \hat{\underset{\sim}{\omega}}_K = {}^R\underset{\sim}{\omega}_J + \hat{\underset{\sim}{\omega}}_K \qquad (10) $$

Regarding the body-connection array, recall that $\mathscr{L}^0(K)=K$, $\mathscr{L}^1(K)=\mathscr{L}(K)$, $\mathscr{L}^2(K)=\mathscr{L}\left(\mathscr{L}(K)\right)$, etc.

Using Eq. (10), the angular velocities of the bodies in the system can be developed starting with the reference body and then radiating outward through the branches of the system. Resolving the components of ${}^R\underset{\sim}{\omega}_K$ and $\hat{\underset{\sim}{\omega}}_K$ in body K and the components of ${}^R\underset{\sim}{\omega}_J$ in body $J=\mathscr{L}(K)$, Eq. (10) can be rewritten in component form as

$$ \left\{{}^R\omega'_K\right\} = \left[{}^JR_K\right]\left\{{}^R\omega'_J\right\} + \left\{\hat{\omega}'_K\right\} \qquad (11) $$

Here, the relative transformation matrix $\left[{}^JR_K\right]$ transforms the components of the angular velocity vector of body J into the body K reference frame.

Eq. (10) can be differentiated to find a ***recursive equation*** for the partial angular velocities as well. These partial derivatives are ***non-zero*** only for $p=1,\ldots,3N$, and they are ***zero*** for $p>3N$.

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_K}{\partial y_p} = \frac{\partial\,{}^R\underset{\sim}{\omega}_J}{\partial y_p} + \frac{\partial \hat{\underset{\sim}{\omega}}_K}{\partial y_p} \qquad (12) $$

Note that the partial derivatives of the relative angular velocity $\hat{\underset{\sim}{\omega}}_K$ are ***non-zero*** only for $p=(3K-2),(3K-1),3K$. So, the partial angular velocities of body K can be calculated as

$$ \frac{\partial\,{}^R\underset{\sim}{\omega}_K}{\partial y_p} = \frac{\partial\,{}^R\underset{\sim}{\omega}_J}{\partial y_p} \quad \left(p=1,\ldots,(3K-3)\right) \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_K}{\partial y_p} = \begin{cases} \underset{\sim}{n}_1 \left(p=3K-2\right) \\ \underset{\sim}{n}_2 \left(p=3K-1\right) \\ \underset{\sim}{n}_3 \left(p=3K\right) \end{cases} \qquad \frac{\partial\,{}^R\underset{\sim}{\omega}_K}{\partial y_p} = \underset{\sim}{0} \quad \left(p>3K\right) \qquad (13) $$

Note (as assumed above) if the bodies are numbered starting with "1" as the reference body and increasing numbers while moving outward along the branches, the ***angular velocity*** of a body will ***not depend*** on the variables associated with ***higher-numbered bodies***.

Eq. (12) can be written in component form as follows.

$$\left[{}^R\omega'_{K,y}\right]_{3\times 6N} = \left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right] + \left[\hat{\omega}'_{K,y}\right] \quad (14)$$

Here, the ***relative transformation matrix*** $\left[{}^J R_K\right]$ transforms the components of the partial angular velocity vectors of body J (which are resolved in the body J frame) into the body K reference frame. The ***partial relative angular velocity matrix*** $\left[\hat{\omega}'_{K,y}\right]$ of body K can be partitioned into "$2N$" 3×3 matrices as follows.

$$\left[\hat{\omega}'_{K,y}\right]_{3\times 6N} = \Big[\underset{1}{[0]},[0],\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]},\underset{N+1}{[0]},[0],\ldots,\underset{2N}{[0]}\Big] = \Big[\underset{1}{[0]},[0],\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]},[0]_{3\times 3N}\Big] \quad (15)$$

All 3×3 partitions are ***zero*** except the one in the K^{th} partition which is the 3×3 ***identity matrix***. Note the partial angular velocity matrix $\left[{}^R\omega'_{J,y}\right]$ and the subsequent product $\left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right]$ can also be partitioned into "$2N$" 3×3 matrices which may be ***non-zero*** in partitions $1 \rightarrow (K-1)$, but are ***zero*** everywhere else. Hence, the matrix summation indicated in Eq. (14) need not actually be done. The partial angular velocity matrix for body K can be formed by ***simply changing*** the K^{th} partition of the product $\left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right]$ to an ***identity matrix***.

Note finally, that the ***body-frame components*** of the angular velocities of the bodies can now be written in terms of the partial angular velocity matrices as follows.

$$\left\{{}^R\omega'_K\right\} = \left[{}^R\omega'_{K,y}\right]_{3\times 6N}\{y\}_{6N\times 1} = \left[\left[{}^R\omega'_{K,y_1}\right]_{3\times 3N} \quad \left[{}^R\omega'_{K,y_2}\right]_{3\times 3N}\right]\begin{Bmatrix}\{y_1\}_{3N\times 1}\\ \{y_2\}_{3N\times 1}\end{Bmatrix} \quad (16)$$

The last part Eq. (16) is written in partitioned form, separating the parts associated with the elements of $\{y_1\}$ from those associated with $\{y_2\}$. Noting that all the elements of $\left[{}^R\omega'_{K,y_2}\right]_{3\times 3N}$ are ***zero***, this equation can be further simplified to give

$$\left\{{}^R\omega'_K\right\} = \left[{}^R\omega'_{K,y_1}\right]\{y_1\} + \left[{}^R\omega'_{K,y_2}\right]\{y_2\} = \left[{}^R\omega'_{K,y_1}\right]\{y_1\} \quad (17)$$

Even this final product can be ***further simplified*** by noting that ***many*** of the ***elements*** of $\left[{}^R\omega'_{K,y_1}\right]$ are also ***zero***, so they may be ignored in the product.

<u>Angular Acceleration</u>

The ***angular accelerations*** of the bodies are found by differentiating the angular velocities either in the ***inertial frame*** or in the ***body frame***.

$${}^R\underset{\sim}{\alpha}_K = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_K\right) = \frac{{}^K d}{dt}\left({}^R\underset{\sim}{\omega}_K\right)$$

The ***body-fixed components*** of ${}^{R}\underset{\sim}{\alpha}_K$ the angular acceleration of body K are found by ***differentiating*** the body-fixed components of the angular velocity of body K in Eq. (17).

$$\boxed{\begin{aligned}\left\{{}^{R}\alpha'_K\right\} &= \left\{{}^{R}\dot{\omega}'_K\right\} \\ &= \left[{}^{R}\omega'_{K,y}\right]\left\{\dot{y}\right\} + \left[{}^{R}\dot{\omega}'_{K,y}\right]\left\{y\right\} \\ &= \left[{}^{R}\omega'_{K,y_1}\right]\left\{\dot{y}_1\right\} + \left[{}^{R}\dot{\omega}'_{K,y_1}\right]\left\{y_1\right\}\end{aligned}}$$

Here,

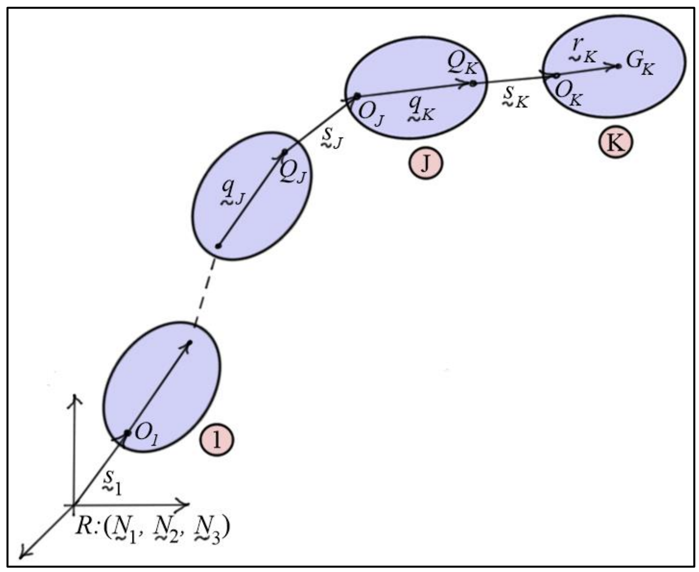
$$\left[{}^{R}\dot{\omega}'_{K,y_1}\right]_{3\times 3N} = \left[{}^{J}R_K\right]\left[{}^{R}\dot{\omega}'_{J,y_1}\right] + \left[{}^{J}\dot{R}_K\right]\left[{}^{R}\omega'_{J,y_1}\right] + \underbrace{\left[\dot{\hat{\omega}}'_{K,y_1}\right]}_{\text{zero}}$$

$$\Rightarrow \boxed{\left[{}^{R}\dot{\omega}'_{K,y_1}\right]_{3\times 3N} = \left[{}^{J}R_K\right]\left[{}^{R}\dot{\omega}'_{J,y_1}\right] + \left[{}^{J}\tilde{\omega}'_K\right]^T\left[{}^{J}R_K\right]\left[{}^{R}\omega'_{J,y_1}\right]} \qquad (18)$$

Eq. (18) provides a ***recursive relationship*** for finding the time derivatives of the partial angular velocity matrices. Recall, from above that $\left[{}^{R}\omega'_{K,y_2}\right]_{3\times 3N}$ is a ***zero*** matrix, so $\left[{}^{R}\dot{\omega}'_{K,y_2}\right]_{3\times 3N}$ is also a ***zero*** matrix.

Mass-Center Position Vectors

Consider a ***typical branch*** of a multibody system as shown in the diagram. Each body K has a ***mass-center*** G_K, an ***origin*** O_K, and a ***reference point*** Q_K. The points G_K and O_K are fixed in body K, and the point Q_K is fixed in the adjacent, lower body J $\left(J = \mathscr{L}(K)\right)$. The point O_K is positioned relative to O_J the origin of body J by the position vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$.



The position vector of O_K relative to the inertial system can be written as

$$\boxed{\underset{\sim}{p}_{O_K} = \underset{\sim}{p}_{O_J} + \underset{\sim}{q}_K + \underset{\sim}{s}_K} \quad \left(K = 1,\ldots,N\right) \qquad (19)$$

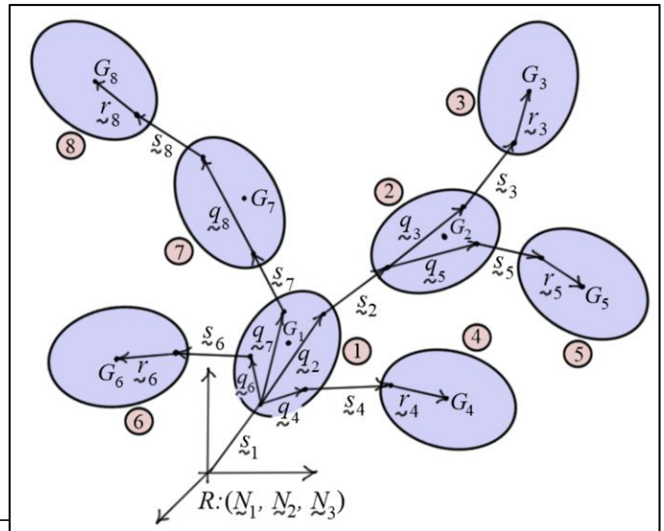
Given that $\underset{\sim}{p}_{O_1} = \underset{\sim}{s}_1 \; \left(\underset{\sim}{q}_1 \triangleq \underset{\sim}{0}\right)$, Eq. (19) is a ***recursive relationship*** that can be used to build the ***position vectors*** of the ***origins*** of all the bodies of the system. The components of $\underset{\sim}{p}_{O_K}$ are resolved in the body K fixed system, but the components of $\underset{\sim}{p}_{O_J}$, $\underset{\sim}{q}_K$, and $\underset{\sim}{s}_K$ are all resolved in body J. So, Eq. (19) can be written in component form as

$$\boxed{\left\{p'_{O_K}\right\} = \left[{}^{J}R_K\right]\left(\left\{p'_{O_J}\right\} + \left\{q'_K\right\} + \left\{s'_K\right\}\right)} \qquad (20)$$

Finally, resolving the components of $\underset{\sim}{r}_K$ in body *K*, the ***body-frame components*** of the ***mass-center position vectors*** can be written as

$$\left\{p'_{G_K}\right\}=\left\{p'_{O_K}\right\}+\left\{r'_K\right\}=\left[{}^J R_K\right]\left(\left\{p'_{O_J}\right\}+\left\{q'_K\right\}+\left\{s'_K\right\}\right)+\left\{r'_K\right\} \qquad (21)$$

Consider now the eight-body example system. Using Eq. (21), the ***position vectors*** of the ***mass-centers*** of the bodies can be written in component form as follows.



$$\begin{aligned}\left\{p'_{G_1}\right\}&=\left\{p'_{O_1}\right\}+\left\{r'_1\right\}\\&=\left[R_1\right]\left\{s'_1\right\}+\left\{r'_1\right\}\end{aligned}$$

$$\begin{aligned}\left\{p'_{G_2}\right\}&=\left[{}^1R_2\right]\left(\left\{p'_{O_1}\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{r'_2\right\}\\&=\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{r'_2\right\}\end{aligned}$$

$$\begin{aligned}\left\{p'_{G_3}\right\}&=\left[{}^2R_3\right]\left(\left\{p'_{O_2}\right\}+\left\{q'_3\right\}+\left\{s'_3\right\}\right)+\left\{r'_3\right\}\\&=\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{q'_3\right\}+\left\{s'_3\right\}\right)+\left\{r'_3\right\}\end{aligned}$$

$$\left\{p'_{G_4}\right\}=\left[{}^1R_4\right]\left(\left\{p'_{O_1}\right\}+\left\{q'_4\right\}+\left\{s'_4\right\}\right)+\left\{r'_4\right\}=\left[{}^1R_4\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_4\right\}+\left\{s'_4\right\}\right)+\left\{r'_4\right\}$$

$$\begin{aligned}\left\{p'_{G_5}\right\}&=\left[{}^2R_5\right]\left(\left\{p'_{O_2}\right\}+\left\{q'_5\right\}+\left\{s'_5\right\}\right)+\left\{r'_5\right\}\\&=\left[{}^2R_5\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{q'_5\right\}+\left\{s'_5\right\}\right)+\left\{r'_5\right\}\end{aligned}$$

$$\left\{p'_{G_6}\right\}=\left[{}^1R_6\right]\left(\left\{p'_{O_1}\right\}+\left\{q'_6\right\}+\left\{s'_6\right\}\right)+\left\{r'_6\right\}=\left[{}^1R_6\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_6\right\}+\left\{s'_6\right\}\right)+\left\{r'_6\right\}$$

$$\left\{p'_{G_7}\right\}=\left[{}^1R_7\right]\left(\left\{p'_{O_1}\right\}+\left\{q'_7\right\}+\left\{s'_7\right\}\right)+\left\{r'_7\right\}=\left[{}^1R_7\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_7\right\}+\left\{s'_7\right\}\right)+\left\{r'_7\right\}$$

$$\begin{aligned}\left\{p'_{G_8}\right\}&=\left[{}^7R_8\right]\left(\left\{p'_{O_7}\right\}+\left\{q'_8\right\}+\left\{s'_8\right\}\right)+\left\{r'_8\right\}\\&=\left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_7\right\}+\left\{s'_7\right\}\right)+\left\{q'_8\right\}+\left\{s'_8\right\}\right)+\left\{r'_8\right\}\end{aligned}$$

<u>Mass-Center Velocities</u>

The velocities of the mass-centers of the bodies can be found by first finding the ***velocities*** of the ***origins*** of the bodies. This can be done as follows.

$${}^R\underset{\sim}{v}_{O_K}=\frac{{}^R d\,\underset{\sim}{p}_{O_K}}{dt}=\frac{{}^R d}{dt}\left(\underset{\sim}{p}_{O_J}+\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)=\frac{{}^R d\,\underset{\sim}{p}_{O_J}}{dt}+\frac{{}^R d}{dt}\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)={}^R\underset{\sim}{v}_{O_J}+\frac{{}^R d}{dt}\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)$$

The last term can be expanded using the derivative rule (that relates the derivatives of a vector in different reference frames) as follows.

$$\frac{{}^R d}{dt}\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)=\frac{{}^J d}{dt}\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)+{}^R\underset{\sim}{\omega}_J\times\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)=\frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right)+{}^R\underset{\sim}{\omega}_J\times\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)$$

Combining these two results gives

$$\boxed{{}^R\underset{\sim}{v}_{O_K}={}^R\underset{\sim}{v}_{O_J}+\frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right)+{}^R\underset{\sim}{\omega}_J\times\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)}\tag{22}$$

Eq. (22) can be written in component form as follows.

$$\boxed{\left\{{}^R v'_{O_K}\right\}=\left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^R v'_{O_J}\right\}+\left\{\dot{s}'_K\right\}+\left[{}^R\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\}+\left\{s'_K\right\}\right\}}\tag{23}$$

Here, $\left\{{}^R v'_{O_J}\right\}$ are components of ${}^R\underset{\sim}{v}_{O_J}$ in body $\mathscr{L}(J)$, and $\left\{{}^R v'_{O_K}\right\}$ are the components of ${}^R\underset{\sim}{v}_{O_K}$ in body $J=\mathscr{L}(K)$. This result allows the velocities of the origins of the bodies to be calculated ***recursively***, starting with the velocity of O_1, the origin of body 1, the reference body.

Given the velocities of the origins of the bodies, the ***velocities*** of the ***mass-centers*** of the bodies can be calculated as follows.

$$\boxed{{}^R\underset{\sim}{v}_{G_K}={}^R\underset{\sim}{v}_{O_K}+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{r}_K\right)}\tag{24}$$

Resolving the components of ${}^R\underset{\sim}{v}_{G_K}$ in body *K*, the above equation can be written in component form as follows.

$$\boxed{\left\{{}^R v'_{G_K}\right\}=\left[{}^J R_K\right]\left\{{}^R v'_{O_K}\right\}+\left[{}^R\tilde{\omega}'_K\right]\left\{r'_K\right\}}\tag{25}$$

<u>Mass-Center Partial Velocities</u>

The ***partial velocities*** of the ***mass centers*** of the bodies can be written in terms of the ***partial velocities*** of the ***origins*** of the bodies. To this end, rewrite Eq. (23) as follows.

$$\begin{aligned}
\left\{{}^R v'_{O_K}\right\}&=\left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^R v'_{O_J}\right\}+\left\{\dot{s}'_K\right\}+\left[{}^R\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\}+\left\{s'_K\right\}\right\}\\
&=\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right]\{y\}-\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]\{y\}+\left[{}^J v'_{O_K,y}\right]\{y\}\\
&=\left(\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right]-\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]+\left[{}^J v'_{O_K,y}\right]\right)\{y\}\\
&\triangleq\left[{}^R v'_{O_K,y}\right]\{y\}
\end{aligned}$$

$$\Rightarrow\boxed{\left[{}^R v'_{O_K,y}\right]=\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right]-\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]+\left[{}^J v'_{O_K,y}\right]}\tag{26}$$

Here, $\left[{}^R v'_{O_J,y}\right]_{3\times 6N}$ and $\left[{}^R v'_{O_K,y}\right]_{3\times 6N}$ are the partial velocity matrices of the origin points O_J and O_K, and $\left[{}^J v'_{O_K,y}\right]$ can be partitioned and defined as follows.

$$\boxed{\left[{}^J v'_{O_K,y}\right]_{3\times 6N}=\left[\left[{}^J v'_{O_K,y_1}\right]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right]=\left[[0]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right]}\tag{27}$$

with

$$\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N} = \left[\underset{1}{[0]},\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]}\right] \tag{28}$$

In Eq. (28), $[0]$ represents the 3×3 ***zero*** matrix, and $[I]$ represents the 3×3 ***identity*** matrix.

Eqs. (26)-(28) provide a means to ***recursively*** calculate the partial velocity matrices of the ***origins*** of the bodies. Using this result, the partial velocity matrices of the ***mass-centers*** of the bodies can be calculated as follows. Returning to Eq. (25), write

$$\begin{aligned}\left\{{}^{R}v'_{G_K}\right\} &= \left[{}^{J}R_K\right]\left\{{}^{R}v'_{O_K}\right\} + \left[{}^{R}\tilde{\omega}'_K\right]\left\{r'_K\right\}\\ &= \left[{}^{J}R_K\right]\left\{{}^{R}v'_{O_K}\right\} - \left[\tilde{r}'_K\right]\left\{{}^{R}\omega'_K\right\}\\ &= \left(\left[{}^{J}R_K\right]\left[{}^{R}v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]\right)\{y\}\\ &\triangleq \left[{}^{R}v'_{G_K,y}\right]\{y\}\end{aligned} \tag{29}$$

$$\Rightarrow \boxed{\left[{}^{R}v'_{G_K,y}\right] = \left[{}^{J}R_K\right]\left[{}^{R}v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]} \tag{30}$$

Mass-Center Accelerations

The ***accelerations*** of the ***mass-centers*** of the bodies can be found by ***differentiating*** the ***velocities*** using the ***derivative rule***. That is,

$$\boxed{\frac{{}^{R}d\,{}^{R}\underset{\sim}{v}_{G_K}}{dt} = \frac{{}^{K}d\,{}^{R}\underset{\sim}{v}_{G_K}}{dt} + \left({}^{R}\underset{\sim}{\omega}_K \times {}^{R}\underset{\sim}{v}_{G_K}\right)}$$

This result can be expressed in component form as

$$\boxed{\begin{aligned}\left\{{}^{R}a'_{G_K}\right\} &= \left\{{}^{R}\dot{v}'_{G_K}\right\} + \left[{}^{R}\tilde{\omega}'_K\right]\left\{{}^{R}v'_{G_K}\right\}\\ &= \left[{}^{R}v'_{G_K,y}\right]\{\dot{y}\} + \left[{}^{R}\dot{v}'_{G_K,y}\right]\{y\} + \left[{}^{R}\tilde{\omega}'_K\right]\left\{{}^{R}v'_{G_K}\right\}\end{aligned}} \quad \text{(all components in body } K\text{)}$$

Using Eq. (30), the ***time derivatives*** of the ***partial velocities*** of the ***mass-centers*** can be written as follows.

$$\boxed{\begin{aligned}\left[{}^{R}\dot{v}'_{G_K,y}\right] &= \left[{}^{J}R_K\right]\left[{}^{R}\dot{v}'_{O_K,y}\right] + \left[{}^{J}\dot{R}_K\right]\left[{}^{R}v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^{R}\dot{\omega}'_{K,y}\right]\\ &= \left[{}^{J}R_K\right]\left[{}^{R}\dot{v}'_{O_K,y}\right] + \left[{}^{J}\tilde{\omega}'_K\right]^T\left[{}^{J}R_K\right]\left[{}^{R}v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^{R}\dot{\omega}'_{K,y}\right]\end{aligned}} \tag{31}$$

Using Eq. (26), the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the bodies can be calculated as follows.

$$\left[{}^{R}\dot{v}'_{O_K,y}\right] = \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^{R}\dot{v}'_{O_J,y}\right] + \left[{}^{\mathscr{L}(J)}\dot{R}_J\right]\left[{}^{R}v'_{O_J,y}\right] - \left[\dot{\tilde{s}}'_K\right]\left[{}^{R}\omega'_{J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^{R}\dot{\omega}'_{J,y}\right] + \underbrace{\left[{}^{J}\dot{v}'_{O_K,y}\right]}_{\text{zero}}$$

or

$$\boxed{\left[{}^{R}\dot{v}'_{O_K,y}\right] = \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^{R}\dot{v}'_{O_J,y}\right] + \left[{}^{\mathscr{L}(J)}\tilde{\omega}'_J\right]^T\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^{R}v'_{O_J,y}\right] - \left[\dot{\tilde{s}}'_K\right]\left[{}^{R}\omega'_{J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^{R}\dot{\omega}'_{J,y}\right]} \tag{32}$$

This result allows the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the bodies to be calculated in terms of the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the ***lower bodies***.

Generalized Forces

Let the forces and torques acting on each body of the system be replaced by an ***equivalent force system*** consisting of a single force $\underset{\sim}{F}_K$ acting at the mass-center G_K and a single moment $\underset{\sim}{M}_K$. Then the ***generalized forces*** for the system can be calculated as

$$\boxed{F_{y_i} = \sum_{K=1}^{N}\left(\left(\underset{\sim}{F}_K \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right) + \left(\underset{\sim}{M}_K \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}\right)\right)} \tag{33}$$

or, in component form, the column vector of generalized forces is

$$\boxed{\left\{F_y\right\}_{6N\times 1} = \sum_{K=1}^{N}\left(\left[{}^R v'_{G_K,y}\right]^T \left\{F'_K\right\} + \left[{}^R\omega'_{K,y}\right]^T \left\{M'_K\right\}\right)} \tag{34}$$

where $\left\{F'_K\right\}$ represents the body K components of the force-vector $\underset{\sim}{F}_K$ and $\left\{M'_K\right\}$ represents the body K components of the moment-vector $\underset{\sim}{M}_K$.

Equations of Motion of the Unconstrained System

Assuming all "$6N$" of the generalized speeds are ***independent***, Kane's equations of motion for the multibody system can be written as

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} = F_{y_i}} \qquad (i = 1,\ldots,6N) \tag{35}$$

Here, the ***generalized forces*** on the right side of the equation are the entries of the generalized force column vector of Eq. (34). The terms on the left side of the equation can be written as follows.

1. $\underset{\sim}{a}_{G_K} \quad \rightarrow \quad \left\{{}^R a'_{G_K}\right\} = \left[{}^R v'_{G_K,y}\right]\left\{\dot{y}\right\} + \left[{}^R \dot{v}'_{G_K,y}\right]\left\{y\right\} + \left[{}^R\tilde{\omega}'_K\right]\left\{{}^R v'_{G_K}\right\}$

2. ${}^R\underset{\sim}{\alpha}_K \quad \rightarrow \quad \left\{{}^R\alpha'_K\right\} = \left\{{}^R\dot{\omega}'_K\right\} = \left[{}^R\omega'_{K,y_1}\right]\left\{\dot{y}_1\right\} + \left[{}^R\dot{\omega}'_{K,y_1}\right]\left\{y_1\right\}$

3. $\displaystyle\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right) \rightarrow$

$$\boxed{\begin{aligned}\sum_{K=1}^{N}\left(m_K \left[{}^R v'_{G_K,y}\right]^T \left\{{}^R a'_{G_K}\right\}\right) = &\sum_{K=1}^{N}\left(m_K \left[{}^R v'_{G_K,y}\right]^T \left[{}^R v'_{G_K,y}\right]\left\{\dot{y}\right\}\right) \\ &+ \sum_{K=1}^{N}\left(m_K \left[{}^R v'_{G_K,y}\right]^T \left[{}^R \dot{v}'_{G_K,y}\right]\left\{y\right\}\right) \\ &+ \sum_{K=1}^{N}\left(m_K \left[{}^R v'_{G_K,y}\right]^T \left[{}^R\tilde{\omega}'_K\right]\left\{{}^R v'_{G_K}\right\}\right)\end{aligned}} \tag{36}$$

4. $\displaystyle\sum_{K=1}^{N}\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \quad \rightarrow$

$$\boxed{\sum_{K=1}^{N}\left(\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left\{{}^{R}\alpha'_{K}\right\}\right)=\sum_{K=1}^{N}\left(\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left(\left[{}^{R}\omega'_{K,y_1}\right]\left\{\dot{y}_1\right\}+\left[{}^{R}\dot{\omega}'_{K,y_1}\right]\left\{y_1\right\}\right)\right)} \tag{37}$$

5. ${}^{R}\underset{\sim}{\omega}_{K}\times \underset{\sim}{H}_{G_K} \quad \rightarrow \quad \left[{}^{R}\tilde{\omega}'_{K}\right]\left[I'_{G_K}\right]\{{}^{R}\omega'_{K}\}$ (body-fixed components)

6. $$\sum_{K=1}^{N}\left({}^{R}\underset{\sim}{\omega}_{K}\times \underset{\sim}{H}_{G_K}\right)\cdot\frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_i} \quad\rightarrow\quad \boxed{\sum_{K=1}^{N}\left[{}^{R}\omega'_{K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_{K}\right]\left[I'_{G_K}\right]\left\{{}^{R}\omega'_{K}\right\}} \tag{38}$$

Substituting from Eqs. (34) and (36)-(38) into Eq. (35) gives

$$\begin{aligned}
&\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}v'_{G_K,y}\right]+\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left[{}^{R}\omega'_{K,y}\right]\right)\{\dot{y}\}\\
&\quad=\sum_{K=1}^{N}\left(\left[{}^{R}v'_{G_K,y}\right]^{T}\left\{F'_{K}\right\}+\left[{}^{R}\omega'_{K,y}\right]^{T}\left\{M'_{K}\right\}\right)-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\dot{v}'_{G_K,y}\right]\{y\}\right)\\
&\qquad-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_{K}\right]\left\{{}^{R}v'_{G_K}\right\}\right)-\sum_{K=1}^{N}\left(\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left[{}^{R}\dot{\omega}'_{K,y}\right]\{y\}\right)\\
&\qquad-\sum_{K=1}^{N}\left[{}^{R}\omega'_{K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_{K}\right]\left[I'_{G_K}\right]\left\{{}^{R}\omega'_{K}\right\}
\end{aligned}$$

The above result can be written in the final matrix form

$$\boxed{[A]\{\dot{y}\}=\{f\}} \qquad ([A] \text{ is called the “}\textbf{\textit{generalized mass matrix}}\text{”}) \tag{39}$$

Here,

$$\boxed{[A]=\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}v'_{G_K,y}\right]+\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left[{}^{R}\omega'_{K,y}\right]\right)} \tag{40}$$

$$\boxed{\begin{aligned}
\{f\}&=\sum_{K=1}^{N}\left(\left[{}^{R}v'_{G_K,y}\right]^{T}\left\{F'_{K}\right\}+\left[{}^{R}\omega'_{K,y}\right]^{T}\left\{M'_{K}\right\}\right)-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\dot{v}'_{G_K,y}\right]\{y\}\right)\\
&\quad-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_{K}\right]\left\{{}^{R}v'_{G_K}\right\}\right)-\sum_{K=1}^{N}\left(\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left[{}^{R}\dot{\omega}'_{K,y}\right]\{y\}\right)\\
&\quad-\sum_{K=1}^{N}\left[{}^{R}\omega'_{K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_{K}\right]\left[I'_{G_K}\right]\left\{{}^{R}\omega'_{K}\right\}
\end{aligned}} \tag{41}$$

Eq. (39) represents “$6N$” ***first-order, ordinary differential equations*** for the “$13N$” variables defined by the system state vectors $\{x\}$ and $\{y\}$ of Eq. (1). To form a ***complete set*** of ***differential equations***, Eq. (39) must be supplemented with Eqs. (42) and (43) which are a set of “$7N$” ***first-order, kinematical differential equations***.

$$\{\dot{x}_1\} = \{\dot{\varepsilon}\} = \begin{Bmatrix} \{\dot{\varepsilon}_1\} \\ \{\dot{\varepsilon}_2\} \\ \{\dot{\varepsilon}_3\} \\ \vdots \\ \{\dot{\varepsilon}_N\} \end{Bmatrix} = \tfrac{1}{2} \begin{bmatrix} [\hat{E}'_1]^T & [0] & [0] & [0] & [0] \\ [0] & [\hat{E}'_2]^T & [0] & \ddots & [0] \\ [0] & [0] & [\hat{E}'_3]^T & \ddots & [0] \\ [0] & \ddots & \ddots & \ddots & [0] \\ [0] & [0] & [0] & [0] & [\hat{E}'_N]^T \end{bmatrix} \begin{Bmatrix} \{\hat{\omega}'_1\} \\ \{\hat{\omega}'_2\} \\ \{\hat{\omega}'_3\} \\ \vdots \\ \{\hat{\omega}'_N\} \end{Bmatrix} = \tfrac{1}{2} [\hat{E}']^T \{\hat{\omega}'\} \tag{42}$$

and

$$\{\dot{x}_2\} = \{y_2\} \tag{43}$$

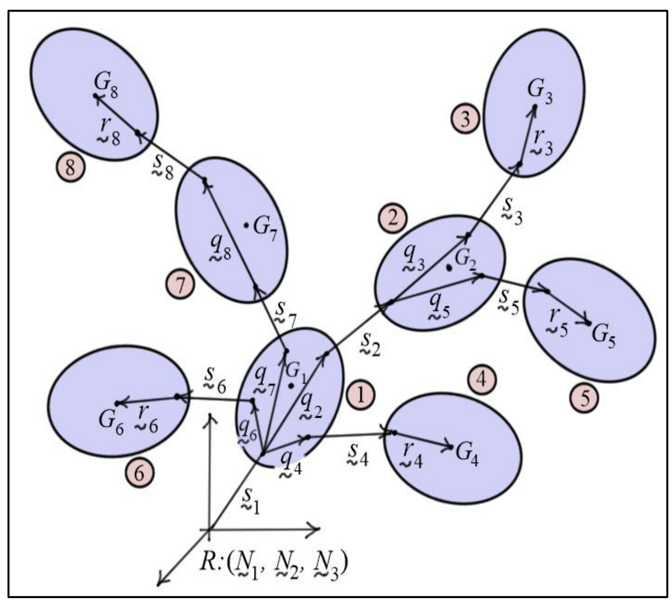
The matrices $[\hat{E}'_K]^T$ that appear on the diagonal of Eq. (42) can be found from Eq. (8).

$$[\hat{E}'_K]^T = \begin{bmatrix} \hat{\varepsilon}_{K4} & -\hat{\varepsilon}_{K3} & \hat{\varepsilon}_{K2} \\ \hat{\varepsilon}_{K3} & \hat{\varepsilon}_{K4} & -\hat{\varepsilon}_{K1} \\ -\hat{\varepsilon}_{K2} & \hat{\varepsilon}_{K1} & \hat{\varepsilon}_{K4} \\ -\hat{\varepsilon}_{K1} & -\hat{\varepsilon}_{K2} & -\hat{\varepsilon}_{K3} \end{bmatrix} \quad (K = 1, \ldots, N) \tag{44}$$

Note that $[\hat{E}'_K]^T$ is formed from $[\hat{E}'_K]^T$ (as defined by Eq. (8)) by removing its ***fourth column***. This allows the angular velocity vectors $\{\hat{\omega}'_K\}$ to be taken as 3×1 vectors and the vector $\{\hat{\omega}'\}$ is a $3N\times 1$ vector. Recall that, for convenience, the angular velocity vector of Eq. (8) was treated as a 4×1 vector whose last element was ***zero*** which made it easy to ***invert*** the equation and solve for the derivatives of the Euler parameters.

Example Eight-Body System

As an ***example*** of how to create the ***kinematical quantities*** required to generate the equations of motion (39)-(41) of a multibody system, consider the eight-body system shown in the diagram. The equations provided above are used below to generate the angular velocities and partial angular velocities of the bodies and their time derivatives and the velocities and partial velocities of the mass centers of the bodies and their time derivatives.



Multibody System with Eight Bodies

Eq. (11) states that the ***angular velocity*** of a body K can be written in terms of the ***angular velocity*** of its ***adjacent***, ***lower body*** J.

$$\boxed{\{{}^R\omega'_K\} = [{}^J R_K]\{{}^R\omega'_J\} + \{\hat{\omega}'_K\}} \quad \boxed{\{\hat{\omega}'_K\} \triangleq \{{}^J\omega'_K\}}$$ (prime indicates body-frame components)

This result allows the ***angular velocities*** to be calculated ***recursively***, starting with the ***reference body*** of the system.

$\left\{ {}^R\omega'_1 \right\} = \left\{ \hat{\omega}'_1 \right\}$ (components in body *1*)

$\left\{ {}^R\omega'_2 \right\} = \left[{}^1R_2 \right] \left\{ {}^R\omega'_1 \right\} + \left\{ \hat{\omega}'_2 \right\}$ (components in body *2*)

$\left\{ {}^R\omega'_3 \right\} = \left[{}^2R_3 \right] \left\{ {}^R\omega'_2 \right\} + \left\{ \hat{\omega}'_3 \right\}$ (components in body *3*)

$\left\{ {}^R\omega'_4 \right\} = \left[{}^1R_4 \right] \left\{ {}^R\omega'_1 \right\} + \left\{ \hat{\omega}'_4 \right\}$ (components in body *4*)

$\left\{ {}^R\omega'_5 \right\} = \left[{}^2R_5 \right] \left\{ {}^R\omega'_2 \right\} + \left\{ \hat{\omega}'_5 \right\}$ (components in body *5*)

$\left\{ {}^R\omega'_6 \right\} = \left[{}^1R_6 \right] \left\{ {}^R\omega'_1 \right\} + \left\{ \hat{\omega}'_6 \right\}$ (components in body *6*)

$\left\{ {}^R\omega'_7 \right\} = \left[{}^1R_7 \right] \left\{ {}^R\omega'_1 \right\} + \left\{ \hat{\omega}'_7 \right\}$ (components in body *7*)

$\left\{ {}^R\omega'_8 \right\} = \left[{}^7R_8 \right] \left\{ {}^R\omega'_7 \right\} + \left\{ \hat{\omega}'_8 \right\}$ (components in body *8*)

Eq. (14) states that the ***partial angular velocity*** of a body K can be written in terms of the ***partial angular velocity*** of its ***adjacent***, ***lower body*** J.

$\left[{}^R\omega'_{K,y} \right]_{3\times 6N} = \left[{}^JR_K \right] \left[{}^R\omega'_{J,y} \right] + \left[\hat{\omega}'_{K,y} \right]$ (prime indicates body-frame components)

Applying this equation to the ***example eight-body system*** gives

$$\left[{}^R\omega'_{1,y} \right]_{3\times 48} = \left[\hat{\omega}'_{1,y} \right] = \left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{2,y} \right]_{3\times 48} = \left[{}^1R_2 \right] \left[{}^R\omega'_{1,y} \right] + \left[\hat{\omega}'_{2,y} \right] = \left[\left[{}^1R_2 \right],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{3,y} \right]_{3\times 48} = \left[{}^2R_3 \right] \left[{}^R\omega'_{2,y} \right] + \left[\hat{\omega}'_{3,y} \right] = \left[\left[{}^2R_3 \right] \left[{}^1R_2 \right], \left[{}^2R_3 \right], [I],[0],[0],[0],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{4,y} \right]_{3\times 48} = \left[{}^1R_4 \right] \left[{}^R\omega'_{1,y} \right] + \left[\hat{\omega}'_{4,y} \right] = \left[\left[{}^1R_4 \right],[0],[0],[I],[0],[0],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{5,y} \right]_{3\times 48} = \left[{}^2R_5 \right] \left[{}^R\omega'_{2,y} \right] + \left[\hat{\omega}'_{5,y} \right] = \left[\left[{}^2R_5 \right] \left[{}^1R_2 \right], \left[{}^2R_5 \right], [0],[0],[I],[0],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{6,y} \right]_{3\times 48} = \left[{}^1R_6 \right] \left[{}^R\omega'_{1,y} \right] + \left[\hat{\omega}'_{6,y} \right] = \left[\left[{}^1R_6 \right],[0],[0],[0],[0],[I],[0],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{7,y} \right]_{3\times 48} = \left[{}^1R_7 \right] \left[{}^R\omega'_{1,y} \right] + \left[\hat{\omega}'_{7,y} \right] = \left[\left[{}^1R_7 \right],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24} \right]$$

$$\left[{}^R\omega'_{8,y} \right]_{3\times 48} = \left[{}^7R_8 \right] \left[{}^R\omega'_{7,y} \right] + \left[\hat{\omega}'_{8,y} \right] = \left[\left[{}^7R_8 \right] \left[{}^1R_7 \right],[0],[0],[0],[0],[0], \left[{}^7R_8 \right],[I],[0]_{3\times 24} \right]$$

Although these results were generated using Eq. (14), they are the ***same*** as those found by simply ***differentiating*** the expressions for the angular velocities.

Eq. (18) states that the ***time-derivative*** of the ***partial angular velocity*** of body K can be written in terms the ***time-derivative*** of the ***partial angular velocity*** of its ***adjacent***, ***lower body*** J as follows.

$$\left[{}^R\dot{\omega}'_{K,y_1}\right]_{3\times 3N} = \left[{}^J R_K\right]\left[{}^R\dot{\omega}'_{J,y_1}\right] + \left[{}^J\tilde{\omega}'_K\right]^T \left[{}^J R_K\right]\left[{}^R\omega'_{J,y_1}\right] \qquad \left[{}^R\dot{\omega}'_{K,y_2}\right]_{3\times 3N} = [0]$$

Applying this equation to the ***example eight-body system*** gives the following.

$$\left[{}^R\dot{\omega}'_{1,y_1}\right]_{3\times 24} = [0]_{3\times 24}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{2,y_1}\right]_{3\times 24} &= \left[{}^1R_2\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_2\right]^T \left[{}^1R_2\right]\left[{}^R\omega'_{1,y_1}\right] = \left[{}^1\tilde{\omega}'_2\right]^T \left[{}^1R_2\right]\left[{}^R\omega'_{1,y_1}\right] \\ &= \left[\left[{}^1\tilde{\omega}'_2\right]^T \left[{}^1R_2\right], [0], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{3,y_1}\right]_{3\times 24} &= \left[{}^2R_3\right]\left[{}^R\dot{\omega}'_{2,y_1}\right] + \left[{}^2\tilde{\omega}'_3\right]^T \left[{}^2R_3\right]\left[{}^R\omega'_{2,y_1}\right] \\ &= \left[{}^2R_3\right]\left[\left[{}^1\tilde{\omega}'_2\right]^T \left[{}^1R_2\right], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad + \left[{}^2\tilde{\omega}'_3\right]^T \left[{}^2R_3\right]\left[\left[{}^1R_2\right], [I], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{4,y_1}\right]_{3\times 24} &= \left[{}^1R_4\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_4\right]^T \left[{}^1R_4\right]\left[{}^R\omega'_{1,y_1}\right] = \left[{}^1\tilde{\omega}'_4\right]^T \left[{}^1R_4\right]\left[{}^R\omega'_{1,y_1}\right] \\ &= \left[\left[{}^1\tilde{\omega}'_4\right]^T \left[{}^1R_4\right], [0], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{5,y_1}\right]_{3\times 24} &= \left[{}^2R_5\right]\left[{}^R\dot{\omega}'_{2,y_1}\right] + \left[{}^2\tilde{\omega}'_5\right]^T \left[{}^2R_5\right]\left[{}^R\omega'_{2,y_1}\right] \\ &= \left[{}^2R_5\right]\left[\left[{}^1\tilde{\omega}'_2\right]^T \left[{}^1R_2\right], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad + \left[{}^2\tilde{\omega}'_5\right]^T \left[{}^2R_5\right]\left[\left[{}^1R_2\right], [I], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{6,y_1}\right]_{3\times 24} &= \left[{}^1R_6\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_6\right]^T \left[{}^1R_6\right]\left[{}^R\omega'_{1,y_1}\right] = \left[{}^1\tilde{\omega}'_6\right]^T \left[{}^1R_6\right]\left[{}^R\omega'_{1,y_1}\right] \\ &= \left[\left[{}^1\tilde{\omega}'_6\right]^T \left[{}^1R_6\right], [0], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{7,y_1}\right]_{3\times 24} &= \left[{}^1R_7\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_7\right]^T \left[{}^1R_7\right]\left[{}^R\omega'_{1,y_1}\right] = \left[{}^1\tilde{\omega}'_7\right]^T \left[{}^1R_7\right]\left[{}^R\omega'_{1,y_1}\right] \\ &= \left[\left[{}^1\tilde{\omega}'_7\right]^T \left[{}^1R_7\right], [0], [0], [0], [0], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{8,y_1}\right]_{3\times 24} &= \left[{}^7R_8\right]\left[{}^R\dot{\omega}'_{7,y_1}\right] + \left[{}^7\tilde{\omega}'_8\right]^T \left[{}^7R_8\right]\left[{}^R\omega'_{7,y_1}\right] \\ &= \left[{}^7R_8\right]\left[\left[{}^1\tilde{\omega}'_7\right]^T \left[{}^1R_7\right], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad + \left[{}^7\tilde{\omega}'_8\right]^T \left[{}^7R_8\right]\left[\left[{}^1R_7\right], [0], [0], [0], [0], [0], [I], [0]\right]\end{aligned}$$

These results are the ***same*** as those found by simply ***differentiating*** the expressions for the partial angular velocities.

Eq. (23) states that the ***velocity*** of the ***origin*** of body K can be written in terms of the ***velocity*** of the ***origin*** of its ***adjacent***, ***lower body*** as follows.

$$\left\{{}^R v'_{O_K}\right\} = \left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^R v'_{O_J}\right\} + \left\{\dot{s}'_K\right\} + \left[{}^R\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\} + \left\{s'_K\right\}\right\}$$

Eq. (25) states that the ***velocity*** of the ***mass-center*** of body K can be written in terms of the ***velocity*** of the ***origin*** of the body as follows.

$$\left\{{}^R v'_{G_K}\right\} = \left[{}^J R_K\right]\left\{{}^R v'_{O_K}\right\} + \left[{}^R\tilde{\omega}'_K\right]\left\{r'_K\right\}$$

Applying these equations to the ***example eight-body system*** gives the following.

$$\left\{{}^R v'_{O_1}\right\} = \left\{\dot{s}'_1\right\}$$

$$\left\{{}^R v'_{G_1}\right\} = \left[R_1\right]\left\{{}^R v'_{O_1}\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{r'_K\right\} = \left[R_1\right]\left\{\dot{s}'_K\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{r'_K\right\}$$

$$\left\{{}^R v'_{O_2}\right\} = \left[R_1\right]\left\{{}^R v'_{O_1}\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\} = \left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}$$

$$\begin{aligned}\left\{{}^R v'_{G_2}\right\} &= \left[{}^1R_2\right]\left\{{}^R v'_{O_2}\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{r'_2\right\} \\ &= \left[{}^1R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_2\right]\left\{r'_2\right\}\end{aligned}$$

$$\begin{aligned}\left\{{}^R v'_{O_3}\right\} &= \left[{}^1R_2\right]\left\{{}^R v'_{O_2}\right\} + \left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\} + \left\{s'_3\right\}\right\} \\ &= \left[{}^1R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}\right) + \left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\} + \left\{s'_3\right\}\right\}\end{aligned}$$

$$\begin{aligned}\left\{{}^R v'_{G_3}\right\} &= \left[{}^2R_3\right]\left\{{}^R v'_{O_3}\right\} + \left[{}^R\tilde{\omega}'_3\right]\left\{r'_3\right\} \\ &= \left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}\right) + \left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\} + \left\{s'_3\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_3\right]\left\{r'_3\right\}\end{aligned}$$

$$\left\{{}^R v'_{O_4}\right\} = \left[R_1\right]\left\{{}^R v'_{O_1}\right\} + \left\{\dot{s}'_4\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\} + \left\{s'_4\right\}\right\} = \left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_4\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\} + \left\{s'_4\right\}\right\}$$

$$\left\{{}^R v'_{G_4}\right\} = \left[{}^1R_4\right]\left\{{}^R v'_{O_4}\right\} + \left[{}^R\tilde{\omega}'_4\right]\left\{r'_4\right\} = \left[{}^1R_4\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_4\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\} + \left\{s'_4\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_4\right]\left\{r'_4\right\}$$

$$\begin{aligned}\left\{{}^R v'_{O_5}\right\} &= \left[{}^1R_2\right]\left\{{}^R v'_{O_2}\right\} + \left\{\dot{s}'_5\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\} + \left\{s'_5\right\}\right\} \\ &= \left[{}^1R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}\right) + \left\{\dot{s}'_5\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\} + \left\{s'_5\right\}\right\}\end{aligned}$$

$$\begin{aligned}\left\{{}^R v'_{G_5}\right\} &= \left[{}^2R_5\right]\left\{{}^R v'_{O_5}\right\} + \left[{}^R\tilde{\omega}'_5\right]\left\{r'_5\right\} \\ &= \left[{}^2R_5\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\} + \left\{s'_2\right\}\right\}\right) + \left\{\dot{s}'_5\right\} + \left[{}^R\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\} + \left\{s'_5\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_5\right]\left\{r'_5\right\}\end{aligned}$$

$$\left\{{}^R v'_{O_6}\right\} = \left[R_1\right]\left\{{}^R v'_{O_1}\right\} + \left\{\dot{s}'_6\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\} + \left\{s'_6\right\}\right\} = \left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_6\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\} + \left\{s'_6\right\}\right\}$$

$$\left\{{}^R v'_{G_6}\right\} = \left[{}^1R_6\right]\left\{{}^R v'_{O_6}\right\} + \left[{}^R\tilde{\omega}'_6\right]\left\{r'_6\right\} = \left[{}^1R_6\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_6\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\} + \left\{s'_6\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_6\right]\left\{r'_6\right\}$$

$$\left\{{}^R v'_{O_7}\right\} = \left[R_1\right]\left\{{}^R v'_{O_1}\right\} + \left\{\dot{s}'_7\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\} = \left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}$$

$$\left\{{}^R v'_{G_7}\right\} = \left[{}^1R_7\right]\left\{{}^R v'_{O_7}\right\} + \left[{}^R\tilde{\omega}'_7\right]\left\{r'_7\right\} = \left[{}^1R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_7\right]\left\{r'_7\right\}$$

$$\begin{aligned}\left\{{}^R v'_{O_8}\right\} &= \left[{}^1R_7\right]\left\{{}^R v'_{O_7}\right\} + \left\{\dot{s}'_8\right\} + \left[{}^R\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\} \\ &= \left[{}^1R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left\{\dot{s}'_8\right\} + \left[{}^R\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\}\end{aligned}$$

$$\begin{aligned}\left\{{}^R v'_{G_8}\right\} &= \left[{}^7R_8\right]\left\{{}^R v'_{O_8}\right\} + \left[{}^R\tilde{\omega}'_8\right]\left\{r'_8\right\} \\ &= \left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^R\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left\{\dot{s}'_8\right\} + \left[{}^R\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\}\right) + \left[{}^R\tilde{\omega}'_8\right]\left\{r'_8\right\}\end{aligned}$$

Eqs. (26)-(28) state that the ***partial velocity*** of the ***origin*** of body K can be written in terms of the ***partial velocity*** of the ***origin*** of its ***adjacent***, ***lower body*** J as

$$\left[{}^R v'_{O_K,y}\right] = \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right] + \left[{}^J v'_{O_K,y}\right]$$

where the last term $\left[{}^J v'_{O_K,y}\right]$ can be ***partitioned*** and ***defined*** as

$$\left[{}^J v'_{O_K,y}\right]_{3\times 6N} = \left[\left[{}^J v'_{O_K,y_1}\right]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right] = \left[\left[0\right]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right]$$

with $\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}$ defined in a partitioned form as follows.

$$\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N} = \Big[\underset{1}{[0]},\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]}\Big]$$

Applying these equations to the ***example eight-body*** system gives the following.

$$\left[{}^R v'_{O_1,y}\right]_{3\times 48} = \left[{}^R v'_{O_1,y}\right] = \left[\left[0\right]_{3\times 24}\left[{}^J v'_{O_1,y_2}\right]_{3\times 24}\right] = \left[\left[0\right]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$\begin{aligned}\left[{}^R v'_{G_1,y}\right]_{3\times 48} &= \left[R_1\right]\left[{}^R v'_{O_1,y}\right] - \left[\tilde{r}'_1\right]\left[{}^R\omega'_{1,y}\right] \\ &= \left[\left[0\right]_{3\times 24},\left[R_1\right],[0],[0],[0],[0],[0],[0],[0]\right] - \left[\left[\tilde{r}'_1\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{O_2,y}\right]_{3\times 48} &= \left[R_1\right]\left[{}^R v'_{O_1,y}\right] - \left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right)\left[{}^R\omega'_{1,y}\right] + \left[{}^1 v'_{O_2,y}\right] \\ &= \left[\left[0\right]_{3\times 24},\left[R_1\right],[0],[0],[0],[0],[0],[0],[0]\right] \\ &\quad - \left[\left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right),[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[\left[0\right]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\end{aligned}$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_2,y}\right]_{3\times 48} &= \left[{}^{1}R_2\right]\left[{}^{R}v'_{O_2,y}\right] - \left[\tilde{r}'_2\right]\left[{}^{R}\omega'_{2,y}\right] \\
&= \left[{}^{1}R_2\right]\left[[0]_{3\times 24},[R_1],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_2\right]\left[\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right),[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[{}^{1}R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_2\right]\left[\left[{}^{1}R_2\right],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{O_3,y}\right]_{3\times 48} &= \left[{}^{1}R_2\right]\left[{}^{R}v'_{O_2,y}\right] - \left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^{R}\omega'_{2,y}\right] + \left[{}^{2}v'_{O_3,y}\right] \\
&= \left[{}^{1}R_2\right]\left[[0]_{3\times 24},[R_1],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_2\right]\left[\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right),[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[{}^{1}R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[\left[{}^{1}R_2\right],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_3,y}\right]_{3\times 48} &= \left[{}^{2}R_3\right]\left[{}^{R}v'_{O_3,y}\right] - \left[\tilde{r}'_3\right]\left[{}^{R}\omega'_{3,y}\right] \\
&= \left[{}^{2}R_3\right]\left[{}^{1}R_2\right]\left[[0]_{3\times 24},[R_1],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{2}R_3\right]\left[{}^{1}R_2\right]\left[\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right),[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[{}^{2}R_3\right]\left[{}^{1}R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{2}R_3\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[\left[{}^{1}R_2\right],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[{}^{2}R_3\right]\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_3\right]\left[\left[{}^{2}R_3\right]\left[{}^{1}R_2\right],\left[{}^{2}R_3\right],\ [I],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{O_4,y}\right]_{3\times 48} &= [R_1]\left[{}^{R}v'_{O_1,y}\right] - \left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^{R}\omega'_{1,y}\right] + \left[{}^{1}v'_{O_4,y}\right] \\
&= [R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[[0]_{3\times 24},[0],[0],[0],[I],[0],[0],[0],[0]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_4,y}\right]_{3\times 48} &= \left[{}^{1}R_4\right]\left[{}^{R}v'_{O_4,y}\right] - \left[\tilde{r}'_4\right]\left[{}^{R}\omega'_{4,y}\right] \\
&= \left[{}^{1}R_4\right][R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_4\right]\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[{}^{1}R_4\right]\left[[0]_{3\times 24},[0],[0],[0],[I],[0],[0],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_4\right]\left[\left[{}^{1}R_4\right],[0],[0],[I],[0],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}
$$

$$\begin{aligned}\left[{}^R v'_{O_5,y}\right]_{3\times 48} &= \left[{}^1R_2\right]\left[{}^R v'_{O_2,y}\right] - \left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[{}^R\omega'_{2,y}\right] + \left[{}^2 v'_{O_5,y}\right] \\ &= \left[{}^1R_2\right]\left[[0]_{3\times 24}, [R_1], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left[{}^1R_2\right]\left[\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right), [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[{}^1R_2\right]\left[[0]_{3\times 24}, [0], [I], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[\left[{}^1R_2\right], [I], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[[0]_{3\times 24}, [0], [0], [0], [0], [I], [0], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{G_5,y}\right]_{3\times 48} &= \left[{}^2R_5\right]\left[{}^R v'_{O_5,y}\right] - \left[\tilde{r}'_5\right]\left[{}^R\omega'_{5,y}\right] \\ &= \left[{}^2R_5\right]\left[{}^1R_2\right]\left[[0]_{3\times 24}, [R_1], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left[{}^2R_5\right]\left[{}^1R_2\right]\left[\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right), [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[{}^2R_5\right]\left[{}^1R_2\right]\left[[0]_{3\times 24}, [0], [I], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left[{}^2R_5\right]\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[\left[{}^1R_2\right], [I], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[{}^2R_5\right]\left[[0]_{3\times 24}, [0], [0], [0], [0], [I], [0], [0], [0]\right] \\ &\quad - \left[\tilde{r}'_5\right]\left[\left[{}^2R_5\right]\left[{}^1R_2\right], \left[{}^2R_5\right], [0], [0], [I], [0], [0], [0], [0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{O_6,y}\right]_{3\times 48} &= \left[R_1\right]\left[{}^R v'_{O_1,y}\right] - \left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[{}^R\omega'_{1,y}\right] + \left[{}^1 v'_{O_6,y}\right] \\ &= \left[R_1\right]\left[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [I], [0], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{G_6,y}\right]_{3\times 48} &= \left[{}^1R_6\right]\left[{}^R v'_{O_6,y}\right] - \left[\tilde{r}'_6\right]\left[{}^R\omega'_{6,y}\right] \\ &= \left[{}^1R_6\right]\left[R_1\right]\left[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left[{}^1R_6\right]\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[{}^1R_6\right]\left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [I], [0], [0]\right] \\ &\quad - \left[\tilde{r}'_6\right]\left[\left[{}^1R_6\right], [0], [0], [0], [0], [I], [0], [0], [0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{O_7,y}\right]_{3\times 48} &= \left[R_1\right]\left[{}^R v'_{O_1,y}\right] - \left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[{}^R\omega'_{1,y}\right] + \left[{}^1 v'_{O_7,y}\right] \\ &= \left[R_1\right]\left[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\right] \\ &\quad - \left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\right] \\ &\quad + \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [I], [0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{G_7,y}\right]_{3\times 48} &= \left[{}^1R_7\right]\left[{}^R v'_{O_7,y}\right] - \left[\tilde{r}'_7\right]\left[{}^R\omega'_{7,y}\right] \\ &= \left[{}^1R_7\right]\left[R_1\right]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\ &\quad -\left[{}^1R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad +\left[{}^1R_7\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad -\left[\tilde{r}'_7\right]\left[\left[{}^1R_7\right],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{O_8,y}\right]_{3\times 48} &= \left[{}^1R_7\right]\left[{}^R v'_{O_7,y}\right] - \left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[{}^R\omega'_{7,y}\right] + \left[{}^7 v'_{O_8,y}\right] \\ &= \left[{}^1R_7\right]\left[R_1\right]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\ &\quad -\left[{}^1R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad +\left[{}^1R_7\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad -\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[\left[{}^1R_7\right],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right] \\ &\quad +\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v'_{G_8,y}\right]_{3\times 48} &= \left[{}^7R_8\right]\left[{}^R v'_{O_8,y}\right] - \left[\tilde{r}'_8\right]\left[{}^R\omega'_{8,y}\right] \\ &= \left[{}^7R_8\right]\left[{}^1R_7\right]\left[R_1\right]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\ &\quad -\left[{}^7R_8\right]\left[{}^1R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad +\left[{}^7R_8\right]\left[{}^1R_7\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad -\left[{}^7R_8\right]\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[\left[{}^1R_7\right],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right] \\ &\quad +\left[{}^7R_8\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right] \\ &\quad -\left[\tilde{r}'_8\right]\left[\left[{}^7R_8\right]\left[{}^1R_7\right],[0],[0],[0],[0],[0],\left[{}^7R_8\right],[I],[0]_{3\times 24}\right]\end{aligned}$$

Finally, Eqs. (31) and (32) state the ***time derivative*** of the ***partial velocity*** of the ***mass-center*** of body K can be written in terms of the ***time derivative*** of the ***partial velocity*** of the ***origin*** of body K, and that the ***time derivative*** of the ***partial velocity*** of the ***origin*** of body K can be written in terms of the ***time derivative*** of the ***partial velocity*** of the ***origin*** of its ***adjacent***, ***lower body***.

$$\begin{aligned}\left[{}^R\dot{v}'_{G_K,y}\right] &= \left[{}^JR_K\right]\left[{}^R\dot{v}'_{O_K,y}\right] + \left[{}^J\dot{R}_K\right]\left[{}^R v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^R\dot{\omega}'_{K,y}\right] \\ &= \left[{}^JR_K\right]\left[{}^R\dot{v}'_{O_K,y}\right] + \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\left[{}^R v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^R\dot{\omega}'_{K,y}\right]\end{aligned}$$

and

$$\begin{aligned}\left[{}^R\dot{v}'_{O_K,y}\right] &= \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R\dot{v}'_{O_J,y}\right] + \left[{}^{\mathscr{L}(J)}\tilde{\omega}'_J\right]^T\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right] - \left[\dot{\tilde{s}}'_K\right]\left[{}^R\omega'_{J,y}\right] \\ &\quad -\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\dot{\omega}'_{J,y}\right]\end{aligned}$$

Applying these equations to the ***example eight-body*** system gives the following results. All the terms on the right side of the equations were defined above. Due to the length of the resulting equations, those results were not substituted into the equations below.

$$\boxed{\left[{}^R\dot{v}'_{O_1,y}\right] = \left[[0]_{3\times 24},[0]_{3\times 24}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_1,y}\right] = \left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right] + \left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right] - \left[\tilde{r}'_1\right]\left[{}^R\dot{\omega}'_{1,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_2,y}\right] = \left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right] + \left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right] - \left[\tilde{\dot{s}}'_2\right]\left[{}^R\omega'_{1,y}\right] - \left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_2,y}\right] = \left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right] + \left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right] - \left[\tilde{r}'_2\right]\left[{}^R\dot{\omega}'_{2,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_3,y}\right] = \left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right] + \left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right] - \left[\tilde{\dot{s}}'_3\right]\left[{}^R\omega'_{2,y}\right] - \left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^R\dot{\omega}'_{2,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_3,y}\right] = \left[{}^2R_3\right]\left[{}^R\dot{v}'_{O_3,y}\right] + \left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[{}^Rv'_{O_3,y}\right] - \left[\tilde{r}'_3\right]\left[{}^R\dot{\omega}'_{3,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_4,y}\right] = \left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right] + \left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right] - \left[\tilde{\dot{s}}'_4\right]\left[{}^R\omega'_{1,y}\right] - \left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_4,y}\right] = \left[{}^1R_4\right]\left[{}^R\dot{v}'_{O_4,y}\right] + \left[{}^1\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]\left[{}^Rv'_{O_4,y}\right] - \left[\tilde{r}'_4\right]\left[{}^R\dot{\omega}'_{4,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_5,y}\right] = \left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right] + \left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right] - \left[\tilde{\dot{s}}'_5\right]\left[{}^R\omega'_{2,y}\right] - \left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[{}^R\dot{\omega}'_{2,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_5,y}\right] = \left[{}^2R_5\right]\left[{}^R\dot{v}'_{O_5,y}\right] + \left[{}^2\tilde{\omega}'_5\right]^T\left[{}^2R_5\right]\left[{}^Rv'_{O_5,y}\right] - \left[\tilde{r}'_5\right]\left[{}^R\dot{\omega}'_{5,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_6,y}\right] = \left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right] + \left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right] - \left[\tilde{\dot{s}}'_6\right]\left[{}^R\omega'_{1,y}\right] - \left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_6,y}\right] = \left[{}^1R_6\right]\left[{}^R\dot{v}'_{O_6,y}\right] + \left[{}^1\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]\left[{}^Rv'_{O_6,y}\right] - \left[\tilde{r}'_6\right]\left[{}^R\dot{\omega}'_{6,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_7,y}\right] = \left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right] + \left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right] - \left[\tilde{\dot{s}}'_7\right]\left[{}^R\omega'_{1,y}\right] - \left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_7,y}\right] = \left[{}^1R_7\right]\left[{}^R\dot{v}'_{O_7,y}\right] + \left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[{}^Rv'_{O_7,y}\right] - \left[\tilde{r}'_7\right]\left[{}^R\dot{\omega}'_{7,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{O_8,y}\right] = \left[{}^1R_7\right]\left[{}^R\dot{v}'_{O_7,y}\right] + \left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[{}^Rv'_{O_7,y}\right] - \left[\tilde{\dot{s}}'_8\right]\left[{}^R\omega'_{7,y}\right] - \left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[{}^R\dot{\omega}'_{7,y}\right]}$$

$$\boxed{\left[{}^R\dot{v}'_{G_8,y}\right] = \left[{}^7R_8\right]\left[{}^R\dot{v}'_{O_8,y}\right] + \left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]\left[{}^Rv'_{O_8,y}\right] - \left[\tilde{r}'_8\right]\left[{}^R\dot{\omega}'_{8,y}\right]}$$