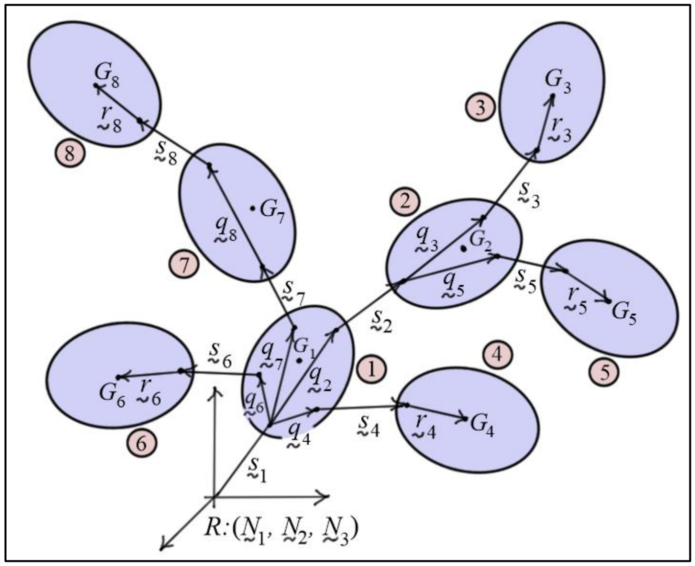
Multibody Dynamics
Equations of Motion for Unconstrained Systems Using Relative Coordinates with Orientation Angles

As mentioned in previous notes, the ***explicit form*** of the equations of motion of a multibody system depends on:

- choice of generalized coordinates
- choice of generalized speeds
- method used to formulate equations
- constraints on system motion

In these notes, ***Kane's equations*** are used to derive the equations of motion of a multibody system using ***relative coordinates*** to describe the ***relative orientation*** and ***relative translation*** between adjoining bodies.



Multibody System with Eight Bodies

Using relative coordinates, the ***kinematic analysis*** is generally ***more complicated***, but the ***constraints*** between adjoining bodies are usually more ***straight-forward*** to formulate. ***Recursive relationships*** can be developed for kinematic variables to streamline the analysis.

In the analysis that follows, the ***relative orientations*** and ***angular velocities*** of bodies are described using ***1-2-3 sequence of body-fixed orientation angles***. As discussed in previous notes, a ***body-connection array*** is used as an aid when developing the kinematic and dynamic equations of motion. For convenience (and without loss of generality), it is assumed herein that the bodies are numbered starting with "1" as the ***reference body*** and ***increasing*** the ***body numbers*** while moving outward along the branches, so a body's ***lower-body*** is also a ***lower-numbered body***.

<u>Generalized Coordinates and Speeds</u>

The generalized coordinates and generalized speeds for a system with "N" bodies are listed below.

- ***Orientation Angles*** $\hat{\theta}_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure the ***orientations*** of the bodies ***relative*** to their ***adjacent, lower bodies***. The orientation angles $\hat{\theta}_{Ki}$ $(i=1,2,3)$ are the 1-2-3 body-fixed orientation angles of body K relative to body $\mathscr{L}(K)$.
- ***Translation variables*** s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure ***displacements*** of the bodies ***relative*** to their ***adjacent, lower bodies***. These variables represent the ***lower-body-frame components*** of the translation vectors of the bodies $(\underset{\sim}{s}_K)$.
- ***Relative angular velocity components*** $\hat{\omega}'_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure the angular velocities of the bodies relative to their ***adjacent, lower bodies***. These are the ***body-frame components*** of the ***relative angular velocity*** vectors of the bodies $\left(\hat{\underset{\sim}{\omega}}_K \triangleq {}^{\mathscr{L}(K)}\underset{\sim}{\omega}_K\right)$.

As described, there are "$6N$" generalized coordinates, $\hat{\theta}_{Ki}\ (K=1,\ldots,N;\ i=1,2,3)$ and $s'_{Ki}\ (K=1,\ldots,N;\ i=1,2,3)$. The "$6N$" generalized speeds are defined to be $\dot{\hat{\theta}}_{Ki}\ (K=1,\ldots,N;\ i=1,2,3)$ and $\dot{s}'_{Ki}\ (K=1,\ldots,N;\ i=1,2,3)$.

System State Vectors

Using the ***generalized coordinates*** and ***speeds*** defined above, the following ***system state vectors*** can be defined.

$$\left\{\hat{\theta}\right\}_{3N\times 1}=[\hat{\theta}_{11},\hat{\theta}_{12},\hat{\theta}_{13},\ldots,\underbrace{\hat{\theta}_{K1},\hat{\theta}_{K2},\hat{\theta}_{K3}}_{\left\{\hat{\theta}_K\right\}^T},\ldots,\hat{\theta}_{N1},\hat{\theta}_{N2},\hat{\theta}_{N3}]^T$$

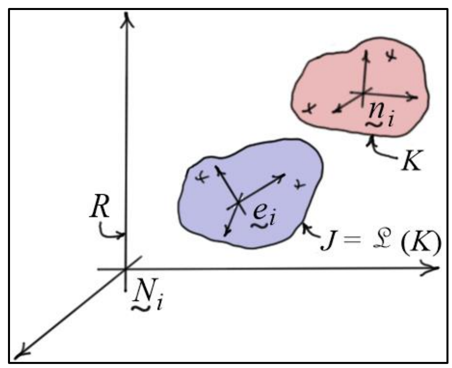
$$\left\{s'\right\}_{3N\times 1}=[s'_{11},s'_{12},s'_{13},\ldots,\underbrace{s'_{K1},s'_{K2},s'_{K3}}_{\left\{s'_K\right\}^T},\ldots,s'_{N1},s'_{N2},s'_{N3}]^T$$

and

$$\boxed{\{x\}_{6N\times 1}=\begin{Bmatrix}\{x_1\}\\\{x_2\}\end{Bmatrix}=\begin{Bmatrix}\left\{\hat{\theta}\right\}_{3N\times 1}\\\{s'\}_{3N\times 1}\end{Bmatrix}}\qquad\boxed{\{y\}_{6N\times 1}=\begin{Bmatrix}\{y_1\}\\\{y_2\}\end{Bmatrix}=\begin{Bmatrix}\{\dot{x}_1\}\\\{\dot{x}_2\}\end{Bmatrix}=\begin{Bmatrix}\left\{\dot{\hat{\theta}}\right\}_{3N\times 1}\\\{\dot{s}'\}_{3N\times 1}\end{Bmatrix}}\tag{1}$$

Transformation Matrices

Consider two bodies of the multibody system. Body J is the adjacent, lower body of body K, that is, $J=\mathscr{L}(K)$. The unit vectors of the two bodies can be written in terms of the inertial frame vectors using the body transformation matrices.



$$\boxed{\{\underset{\sim}{e}\}=[R_J]\{\underset{\sim}{N}\}}\qquad\boxed{\{\underset{\sim}{n}\}=[R_K]\{\underset{\sim}{N}\}}$$

Using these two results, the unit vectors in body K can be written in terms of the unit vectors of body J as follows

$$\boxed{\{\underset{\sim}{n}\}=[R_K]\{\underset{\sim}{N}\}=[R_K][R_J]^T\{\underset{\sim}{e}\}\triangleq\left[{}^JR_K\right]\{\underset{\sim}{e}\}}\tag{2}$$

where $\left[{}^JR_K\right]$ is the relative transformation matrix used to write the unit vectors of body K in terms of the unit vectors of body J.

$$\boxed{\left[{}^JR_K\right]\triangleq[R_K][R_J]^T}$$

Or, ***multiplying*** both sides of the equation on the ***right*** by $[R_J]$ gives

$$\boxed{[R_K]=\left[{}^JR_K\right][R_J]}\tag{3}$$

The relative transformation matrix $\left[{}^JR_K\right]$ can be written in terms of the 1-2-3 orientation angles as follows.

$$\left[{}^{J}R_{K}\right]=\begin{bmatrix} C_{K2}C_{K3} & C_{K1}S_{K3}+S_{K1}S_{K2}C_{K3} & S_{K1}S_{K3}-C_{K1}S_{K2}C_{K3} \\ -C_{K2}S_{K3} & C_{K1}C_{K3}-S_{K1}S_{K2}S_{K3} & S_{K1}C_{K3}+C_{K1}S_{K2}S_{K3} \\ S_{K2} & -S_{K1}C_{K2} & C_{K1}C_{K2} \end{bmatrix} \tag{4}$$

Here, S_{Ki} and C_{Ki} represent the sines and cosines of the relative orientation angles $\hat{\theta}_{Ki}\ (i=1,2,3)$.

The result in Eq. (3) is ***easily extended*** to include as ***many bodies*** as necessary to move from a ***body-frame*** to a ***fixed-frame through*** frames of a ***series*** of ***interconnected bodies***.

$$\left[R_{K}\right]=\left[{}^{\mathscr{L}(K)}R_{K}\right]\left[{}^{\mathscr{L}^2(K)}R_{\mathscr{L}(K)}\right]\cdots\left[{}^{\mathscr{L}^{u_K}(K)}R_{\mathscr{L}^{u_K-1}(K)}\right]\left[R_{\mathscr{L}^{u_K}(K)}\right] \tag{5}$$

Recall that $\mathscr{L}^{u_K}(K)=1$ refers to the ***reference body*** of the system.

<u>Time-Derivatives of the Transformation Matrices</u>

In previous notes, the ***time derivatives*** of the ***transformation matrices*** from the bodies to the ***inertial frame*** were written in terms of ***skew-symmetric matrices*** formed from the components of the ***angular velocities*** of the bodies ***relative*** to the ***fixed frame***. Recall that the "prime" indicates the use of ***body-frame components***.

$$\left[\dot{R}_{K}\right]=\left[\tilde{\omega}'_{K}\right]^{T}\left[R_{K}\right] \quad \text{(components of } {}^{R}\underset{\sim}{\omega}_{K} \text{ are resolved in body } K\text{)} \tag{6}$$

The ***time derivatives*** of the ***transformation matrices between bodies*** in the system were written in terms of ***skew-symmetric matrices*** formed from components of the ***angular velocities*** of the bodies ***relative*** to their ***adjacent, lower bodies***. In the formula that follows, body *J* is the ***lower body*** of body *K*, and the "prime" indicates components resolved in body *K*.

$$\left[{}^{J}\dot{R}_{K}\right]=\left[{}^{J}\tilde{\omega}'_{K}\right]^{T}\left[{}^{J}R_{K}\right] \quad \text{(components of } {}^{J}\underset{\sim}{\omega}_{K} \text{ are resolved in body } K\text{)} \tag{7}$$

<u>Relative Angular Velocity Components and 1-2-3 Orientation Angles</u>

As noted above, the ***angular velocity*** of body K relative to its lower body *J* can be resolved into components in body *K* and written as

$$\underset{\sim}{\hat{\omega}}_{K} \triangleq {}^{J}\underset{\sim}{\omega}_{K}=\hat{\omega}'_{K1}\,\underset{\sim}{n}_{1}+\hat{\omega}'_{K2}\,\underset{\sim}{n}_{2}+\hat{\omega}'_{K3}\,\underset{\sim}{n}_{3}$$

The components $\hat{\omega}'_{Ki}\ (i=1,2,3)$ can be written in terms of the relative orientation angles and their derivatives as

$$\begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix}=\begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \tag{8}$$

Eq. (8) can easily be inverted to give

$$\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} = \begin{bmatrix} C_{K3}/C_{K2} & -S_{K3}/C_{K2} & 0 \\ S_{K3} & C_{K3} & 0 \\ -S_{K2}C_{K3}/C_{K2} & S_{K2}S_{K3}/C_{K2} & 1 \end{bmatrix} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix} \tag{9}$$

As observed in previous notes, Eq. (9) is singular when $\hat{\theta}_{K2} = \pi/2$.

Angular Velocity and Partial Angular Velocity

The angular velocity of body *K* relative to the ground (inertial) frame can be found using the summation rule for angular velocities and the body-connection array.

$$^{R}\underset{\sim}{\omega}_{K} = \underset{\sim}{\hat{\omega}}_{1} + \underset{\sim}{\hat{\omega}}_{\mathscr{L}^{u_K-1}(K)} + \cdots + \underset{\sim}{\hat{\omega}}_{\mathscr{L}(K)} + \underset{\sim}{\hat{\omega}}_{K} = \sum_{i=u_K}^{0} \underset{\sim}{\hat{\omega}}_{\mathscr{L}^{i}(K)} = \sum_{i=u_K}^{1} \underset{\sim}{\hat{\omega}}_{\mathscr{L}^{i}(K)} + \underset{\sim}{\hat{\omega}}_{K} = {}^{R}\underset{\sim}{\omega}_{J} + \underset{\sim}{\hat{\omega}}_{K} \tag{10}$$

Regarding the body-connection array, recall that $\mathscr{L}^0(K) = K$, $\mathscr{L}^1(K) = \mathscr{L}(K)$, $\mathscr{L}^2(K) = \mathscr{L}(\mathscr{L}(K))$, etc., and $\mathscr{L}^{u_K}(K) = 1$.

Using Eq. (10), the angular velocities of the bodies in the system can be developed starting with the reference body and then radiating outward through the branches of the system. Resolving the components of $^{R}\underset{\sim}{\omega}_{K}$ and $\underset{\sim}{\hat{\omega}}_{K}$ in body *K* and the components of $^{R}\underset{\sim}{\omega}_{J}$ in body $J = \mathscr{L}(K)$, Eq. (10) can be rewritten in component form as

$$\left\{ {}^{R}\omega'_{K} \right\} = \left[{}^{J}R_{K} \right] \left\{ {}^{R}\omega'_{J} \right\} + \left\{ \hat{\omega}'_{K} \right\} \tag{11}$$

Here, the relative transformation matrix $\left[{}^{J}R_{K} \right]$ transforms the components of the angular velocity vector of body *J* into the body *K* reference frame.

Eq. (10) can be differentiated to find a ***recursive relationship*** for the partial angular velocities as well. These partial derivatives are ***non-zero*** only for $p = 1, \ldots, 3N$, and they are ***zero*** for $p > 3N$.

$$\frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{J}}{\partial y_p} + \frac{\partial \underset{\sim}{\hat{\omega}}_{K}}{\partial y_p} \tag{12}$$

Note that the partial derivatives of the ***relative angular velocity*** $\underset{\sim}{\hat{\omega}}_{K}$ are ***non-zero*** only for $p = (3K-2), (3K-1), 3K$. So, the partial angular velocities of body *K* can be calculated as follows.

$$\frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{J}}{\partial y_p} \quad \text{for } p = 1, \ldots, (3K-3) \qquad \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = 0 \quad \text{for } p > 3K \tag{13}$$

$$\frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = \begin{Bmatrix} C_{K2}C_{K3} \\ -C_{K2}S_{K3} \\ S_{K2} \end{Bmatrix} \text{ for } p = 3K-2 \qquad \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = \begin{Bmatrix} S_{K3} \\ C_{K3} \\ 0 \end{Bmatrix} \text{ for } p = 3K-1 \qquad \frac{\partial\, {}^{R}\underset{\sim}{\omega}_{K}}{\partial y_p} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \text{ for } p = 3K \tag{14}$$

The components listed in Eq. (14) are components of the partial angular velocities in the body K reference frame. Note that the ***angular velocity*** of a body will ***not depend*** on the variables associated with ***higher-numbered bodies***.

Eq. (12) can be written in component form as follows.

$$\boxed{\left[{}^R\omega'_{K,y}\right]_{3\times 6N} = \left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right] + \left[\hat{\omega}'_{K,y}\right]} \tag{15}$$

Here, the ***relative transformation matrix*** $\left[{}^J R_K\right]$ transforms the components of the partial angular velocity vectors of body J (which are resolved in the body J frame) into the body K reference frame. The ***partial relative angular velocity matrix*** $\left[\hat{\omega}'_{K,y}\right]$ of body K can be partitioned into "$2N$" 3×3 matrices as follows.

$$\boxed{\left[\hat{\omega}'_{K,y}\right]_{3\times 6N} = \Big[\underset{1}{[0]},[0],\ldots,\underset{K-1}{[0]},\underset{K}{[\hat{P}_K^{AV}]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]},\underset{N+1}{[0]},[0],\ldots,\underset{2N}{[0]}\Big]} \tag{16}$$

All 3×3 partitions are ***zero*** except the one in the K^{th} partition which is defined as

$$\boxed{\left[\hat{P}_K^{AV}\right] \triangleq \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}} \tag{17}$$

This is the partial angular velocity matrix for body K motion relative to its lower body J.

Note that the partial angular velocity matrix $\left[{}^R\omega'_{J,y}\right]$ and the subsequent product $\left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right]$ can also be partitioned into "$2N$" 3×3 matrices which may be ***non-zero*** in partitions $1\rightarrow(K-1)$, but are ***zero*** everywhere else. Hence, the matrix summation indicated in Eq. (15) need not actually be done. The partial angular velocity matrix for body K can be formed by ***simply changing*** the K^{th} partition of the product $\left[{}^J R_K\right]\left[{}^R\omega'_{J,y}\right]$ to $\left[\hat{P}_K^{AV}\right]$.

Note finally, that the ***body-frame components*** of the angular velocities of the bodies can now be written in terms of the partial angular velocity matrices as follows.

$$\boxed{\left\{{}^R\omega'_K\right\} = \left[{}^R\omega'_{K,y}\right]_{3\times 6N}\{y\}_{6N\times 1} = \Big[\left[{}^R\omega'_{K,y_1}\right]_{3\times 3N} \quad \left[{}^R\omega'_{K,y_2}\right]_{3\times 3N}\Big]\begin{Bmatrix}\{y_1\}_{3N\times 1}\\ \{y_2\}_{3N\times 1}\end{Bmatrix}} \tag{18}$$

The last part Eq. (18) is written in ***partitioned form***, separating the parts associated with the elements of $\{y_1\}$ from those associated with $\{y_2\}$. Noting that all the elements of $\left[{}^R\omega'_{K,y_2}\right]_{3\times 3N}$ are ***zero***, this equation can be further simplified to give

$$\boxed{\left\{{}^R\omega'_K\right\} = \left[{}^R\omega'_{K,y_1}\right]\{y_1\} + \left[{}^R\omega'_{K,y_2}\right]\{y_2\} = \left[{}^R\omega'_{K,y_1}\right]\{y_1\}} \tag{19}$$

Even this final product can be ***further simplified*** by noting that ***many*** of the ***elements*** of $\left[{}^{R}\omega'_{K,y_1} \right]$ are also ***zero***, so they may be ignored in the product.

Angular Acceleration

The ***angular accelerations*** of the bodies are found by differentiating the angular velocities either in the ***inertial frame*** or in the ***body frame***.

$$
{}^{R}\underset{\sim}{\alpha}_K = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_K \right) = \frac{{}^{K}d}{dt}\left({}^{R}\underset{\sim}{\omega}_K \right)
$$

The ***body-fixed components*** of ${}^{R}\underset{\sim}{\alpha}_K$ the angular acceleration of body K are found by ***differentiating*** the body-fixed components of the angular velocity of body K in Eq. (19).

$$
\begin{aligned}
\left\{ {}^{R}\alpha'_K \right\} &= \left\{ {}^{R}\dot{\omega}'_K \right\} \\
&= \left[{}^{R}\omega'_{K,y} \right]\left\{ \dot{y} \right\} + \left[{}^{R}\dot{\omega}'_{K,y} \right]\left\{ y \right\} \\
&= \left[{}^{R}\omega'_{K,y_1} \right]\left\{ \dot{y}_1 \right\} + \left[{}^{R}\dot{\omega}'_{K,y_1} \right]\left\{ y_1 \right\}
\end{aligned}
$$

Here,

$$
\begin{aligned}
\left[{}^{R}\dot{\omega}'_{K,y_1} \right]_{3\times 3N} &= \left[{}^{J}R_K \right]\left[{}^{R}\dot{\omega}'_{J,y_1} \right] + \left[{}^{J}\dot{R}_K \right]\left[{}^{R}\omega'_{J,y_1} \right] + \left[\dot{\hat{\omega}}'_{K,y_1} \right] \\
&= \left[{}^{J}R_K \right]\left[{}^{R}\dot{\omega}'_{J,y_1} \right] + \left[{}^{J}\tilde{\omega}'_K \right]^T \left[{}^{J}R_K \right]\left[{}^{R}\omega'_{J,y_1} \right] + \left[\dot{\hat{\omega}}'_{K,y_1} \right]
\end{aligned}
\qquad (20)
$$

with

$$
\left[\dot{\hat{\omega}}'_{K,y_1} \right]_{3\times 3N} = \left[\underset{1}{[0]},[0],\ldots,\underset{K-1}{[0]},\underset{K}{[\dot{\hat{P}}_K^{AV}]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]} \right]
$$

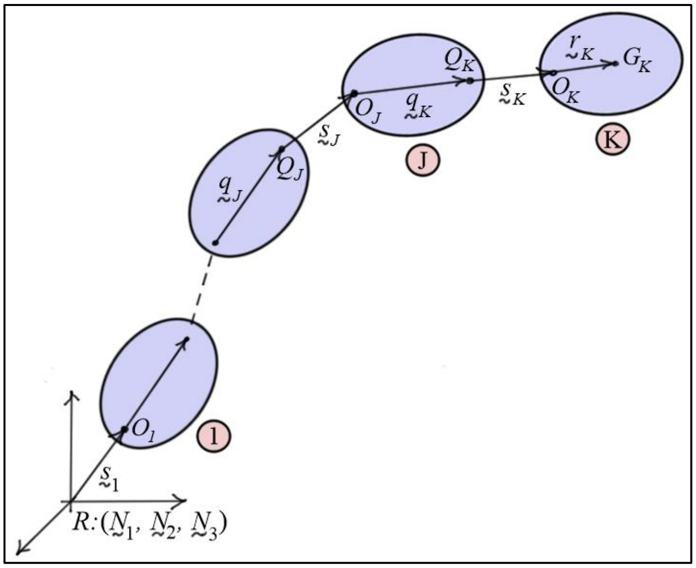
and

$$
\left[\dot{\hat{P}}_K^{AV} \right] = \frac{d}{dt}\left[\hat{P}_K^{AV} \right] = \begin{bmatrix} -\left(S_{K2}C_{K3}\dot{\hat{\theta}}_{K2} + C_{K2}S_{K3}\dot{\hat{\theta}}_{K3} \right) & C_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ \left(S_{K2}S_{K3}\dot{\hat{\theta}}_{K2} - C_{K2}C_{K3}\dot{\hat{\theta}}_{K3} \right) & -S_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ C_{K2}\dot{\hat{\theta}}_{K2} & 0 & 0 \end{bmatrix}
$$

Eq. (20) provides a ***recursive relationship*** for finding the time derivatives of the partial angular velocity matrices. Recall, from above that $\left[{}^{R}\omega'_{K,y_2} \right]_{3\times 3N}$ is a ***zero*** matrix, so $\left[{}^{R}\dot{\omega}'_{K,y_2} \right]_{3\times 3N}$ is also a ***zero*** matrix.

Mass-Center Position Vectors

Consider a ***typical branch*** of a multibody system as shown in the diagram. Each body K has a ***mass-center*** G_K, an ***origin*** O_K, and a ***reference point*** Q_K. The points G_K and O_K are fixed in body K, and the point Q_K is fixed in the adjacent, lower body J $\left(J=\mathscr{L}(K)\right)$. The point O_K is positioned relative to O_J the origin of body J by the position vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$.



The position vector of O_K relative to the inertial system can be written as

$$\boxed{\underset{\sim}{p}_{O_K}=\underset{\sim}{p}_{O_J}+\underset{\sim}{q}_K+\underset{\sim}{s}_K} \quad \left(K=1,\ldots,N\right) \qquad (21)$$

Given that $\underset{\sim}{p}_{O_1}=\underset{\sim}{s}_1 \;\left(\underset{\sim}{q}_1 \triangleq \underset{\sim}{0}\right)$, Eq. (21) is a ***recursive relationship*** that can be used to build the ***position vectors*** of the ***origins*** of all the bodies of the system. The components of $\underset{\sim}{p}_{O_K}$ are resolved in the body K fixed system, but the components of $\underset{\sim}{p}_{O_J}$, $\underset{\sim}{q}_K$, and $\underset{\sim}{s}_K$ are all resolved in body J. So, Eq. (21) can be written in component form as

$$\boxed{\left\{p'_{O_K}\right\}=\left[{}^J R_K\right]\left(\left\{p'_{O_J}\right\}+\left\{q'_K\right\}+\left\{s'_K\right\}\right)} \qquad (22)$$

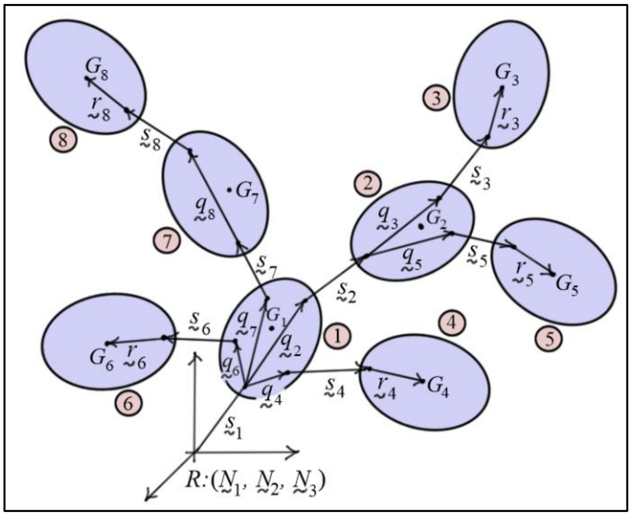
Here, the matrix $\left[{}^J R_K\right]$ transforms the body J components into body K components. Finally, resolving the components of $\underset{\sim}{r}_K$ in body K, the ***body-frame components*** of the ***mass-center position vectors*** can be written as

$$\boxed{\left\{p'_{G_K}\right\}=\left\{p'_{O_K}\right\}+\left\{r'_K\right\}=\left[{}^J R_K\right]\left(\left\{p'_{O_J}\right\}+\left\{q'_K\right\}+\left\{s'_K\right\}\right)+\left\{r'_K\right\}} \qquad (23)$$

Consider now the eight-body example system. Using Eq. (23), the ***position vectors*** of the ***mass-centers*** of the bodies can be written as follows.

$$\boxed{\left\{p'_{G_1}\right\}=\left\{p'_{O_1}\right\}+\left\{r'_1\right\}=\left[R_1\right]\left\{s'_1\right\}+\left\{r'_1\right\}}$$

$$\boxed{\begin{aligned}\left\{p'_{G_2}\right\}&=\left[{}^1R_2\right]\left(\left\{p'_{O_1}\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{r'_2\right\}\\&=\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\}+\left\{q'_2\right\}+\left\{s'_2\right\}\right)+\left\{r'_2\right\}\end{aligned}}$$



$$\left\{p'_{G_3}\right\} = \left[{}^2R_3\right]\left(\left\{p'_{O_2}\right\} + \left\{q'_3\right\} + \left\{s'_3\right\}\right) + \left\{r'_3\right\}$$
$$= \left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left\{q'_3\right\} + \left\{s'_3\right\}\right) + \left\{r'_3\right\}$$

$$\left\{p'_{G_4}\right\} = \left[{}^1R_4\right]\left(\left\{p'_{O_1}\right\} + \left\{q'_4\right\} + \left\{s'_4\right\}\right) + \left\{r'_4\right\} = \left[{}^1R_4\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_4\right\} + \left\{s'_4\right\}\right) + \left\{r'_4\right\}$$

$$\left\{p'_{G_5}\right\} = \left[{}^2R_5\right]\left(\left\{p'_{O_2}\right\} + \left\{q'_5\right\} + \left\{s'_5\right\}\right) + \left\{r'_5\right\}$$
$$= \left[{}^2R_5\right]\left(\left[{}^1R_2\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left\{q'_5\right\} + \left\{s'_5\right\}\right) + \left\{r'_5\right\}$$

$$\left\{p'_{G_6}\right\} = \left[{}^1R_6\right]\left(\left\{p'_{O_1}\right\} + \left\{q'_6\right\} + \left\{s'_6\right\}\right) + \left\{r'_6\right\} = \left[{}^1R_6\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_6\right\} + \left\{s'_6\right\}\right) + \left\{r'_6\right\}$$

$$\left\{p'_{G_7}\right\} = \left[{}^1R_7\right]\left(\left\{p'_{O_1}\right\} + \left\{q'_7\right\} + \left\{s'_7\right\}\right) + \left\{r'_7\right\} = \left[{}^1R_7\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_7\right\} + \left\{s'_7\right\}\right) + \left\{r'_7\right\}$$

$$\left\{p'_{G_8}\right\} = \left[{}^7R_8\right]\left(\left\{p'_{O_7}\right\} + \left\{q'_8\right\} + \left\{s'_8\right\}\right) + \left\{r'_8\right\}$$
$$= \left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left(\left[R_1\right]\left\{s'_1\right\} + \left\{q'_7\right\} + \left\{s'_7\right\}\right) + \left\{q'_8\right\} + \left\{s'_8\right\}\right) + \left\{r'_8\right\}$$

Mass-Center Velocities

The velocities of the mass-centers of the bodies can be found by first finding the ***velocities*** of the ***origins*** of the bodies. This can be done as follows.

$${}^R\underset{\sim}{v}_{O_K} = \frac{{}^Rd\,\underset{\sim}{p}_{O_K}}{dt} = \frac{{}^Rd}{dt}\left(\underset{\sim}{p}_{O_J} + \underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{{}^Rd\,\underset{\sim}{p}_{O_J}}{dt} + \frac{{}^Rd}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = {}^R\underset{\sim}{v}_{O_J} + \frac{{}^Rd}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)$$

The last term can be expanded using the derivative rule that relates the derivative of a vector in reference frame R to the derivative of that vector a body frame as follows.

$$\frac{{}^Rd}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{{}^Jd}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{{}^Jd}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)$$

Combining these two results gives

$${}^R\underset{\sim}{v}_{O_K} = {}^R\underset{\sim}{v}_{O_J} + \frac{{}^Jd}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \tag{24}$$

Eq. (24) can be written in component form as follows.

$$\left\{{}^Rv'_{O_K}\right\} = \left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^Rv'_{O_J}\right\} + \left\{\dot{s}'_K\right\} + \left[{}^R\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\} + \left\{s'_K\right\}\right\} \tag{25}$$

Here, $\left\{{}^Rv'_{O_J}\right\}$ are components of ${}^R\underset{\sim}{v}_{O_J}$ in body $\mathscr{L}(J)$, and $\left\{{}^Rv'_{O_K}\right\}$ are the components of ${}^R\underset{\sim}{v}_{O_K}$ in body $J = \mathscr{L}(K)$. This result allows the velocities of the origins of the bodies to be calculated ***recursively***, starting with the velocity of O_1, the origin of body 1, the reference body.

Given the velocities of the origins of the bodies, the ***velocities*** of the ***mass-centers*** of the bodies can be calculated as follows.

$$\boxed{{}^R\underset{\sim}{v}_{G_K} = {}^R\underset{\sim}{v}_{O_K} + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K\right)} \tag{26}$$

Resolving the components of ${}^R\underset{\sim}{v}_{G_K}$ in body K, the above equation can be written in component form as follows.

$$\boxed{\left\{{}^R v'_{G_K}\right\} = \left[{}^J R_K\right]\left\{{}^R v'_{O_K}\right\} + \left[{}^R\tilde{\omega}'_K\right]\left\{r'_K\right\}} \tag{27}$$

<u>Mass-Center Partial Velocities</u>

The ***partial velocities*** of the ***mass centers*** of the bodies can be written in terms of the ***partial velocities*** of the ***origins*** of the bodies. To this end, rewrite Eq. (25) as follows.

$$\begin{aligned}
\left\{{}^R v'_{O_K}\right\} &= \left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^R v'_{O_J}\right\} + \left\{\dot{s}'_K\right\} + \left[{}^R\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\} + \left\{s'_K\right\}\right\} \\
&= \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right]\{y\} - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]\{y\} + \left[{}^J v'_{O_K,y}\right]\{y\} \\
&= \left(\left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right] + \left[{}^J v'_{O_K,y}\right]\right)\{y\} \\
&\triangleq \left[{}^R v'_{O_K,y}\right]\{y\}
\end{aligned}$$

$$\Rightarrow \boxed{\left[{}^R v'_{O_K,y}\right] = \left[{}^{\mathscr{L}(J)}R_J\right]\left[{}^R v'_{O_J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right] + \left[{}^J v'_{O_K,y}\right]} \tag{28}$$

Here, $\left[{}^R v'_{O_J,y}\right]_{3\times 6N}$ and $\left[{}^R v'_{O_K,y}\right]_{3\times 6N}$ are the partial velocity matrices of the origin points O_J and O_K, and $\left[{}^J v'_{O_K,y}\right]$ can be partitioned and defined as follows.

$$\boxed{\left[{}^J v'_{O_K,y}\right]_{3\times 6N} = \left[\left[{}^J v'_{O_K,y_1}\right]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right] = \left[[0]_{3\times 3N}\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N}\right]} \tag{29}$$

with

$$\boxed{\left[{}^J v'_{O_K,y_2}\right]_{3\times 3N} = \Big[\underset{1}{[0]},\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]}\Big]} \tag{30}$$

In Eq. (30), $[0]$ represents the 3×3 ***zero*** matrix, and $[I]$ represents the 3×3 ***identity*** matrix.

Eqs. (28)-(30) provide a means to ***recursively*** calculate the partial velocity matrices of the ***origins*** of the bodies. Using this result, the partial velocity matrices of the ***mass-centers*** of the bodies can be calculated as follows. Returning to Eq. (27), write

$$\begin{aligned}
\left\{{}^R v'_{G_K}\right\} &= \left[{}^J R_K\right]\left\{{}^R v'_{O_K}\right\} + \left[{}^R\tilde{\omega}'_K\right]\left\{r'_K\right\} = \left[{}^J R_K\right]\left\{{}^R v'_{O_K}\right\} - \left[\tilde{r}'_K\right]\left\{{}^R\omega'_K\right\} \\
&= \left(\left[{}^J R_K\right]\left[{}^R v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^R\omega'_{K,y}\right]\right)\{y\} \\
&\triangleq \left[{}^R v'_{G_K,y}\right]\{y\}
\end{aligned} \tag{31}$$

$$\Rightarrow \boxed{\left[{}^R v'_{G_K,y} \right] = \left[{}^J R_K \right]\left[{}^R v'_{O_K,y} \right] - \left[\tilde{r}'_K \right]\left[{}^R \omega'_{K,y} \right]} \tag{32}$$

Mass-Center Accelerations

The ***accelerations*** of the ***mass-centers*** of the bodies can be found by ***differentiating*** the ***velocities*** using the ***derivative rule***. That is,

$$\boxed{\frac{{}^R d\, {}^R \underset{\sim}{v}_{G_K}}{dt} = \frac{{}^K d\, {}^R \underset{\sim}{v}_{G_K}}{dt} + \left({}^R \underset{\sim}{\omega}_K \times {}^R \underset{\sim}{v}_{G_K} \right)}$$

This result can be expressed in component form as

$$\boxed{\begin{aligned} \left\{ {}^R a'_{G_K} \right\} &= \left\{ {}^R \dot{v}'_{G_K} \right\} + \left[{}^R \tilde{\omega}'_K \right]\left\{ {}^R v'_{G_K} \right\} \\ &= \left[{}^R v'_{G_K,y} \right]\left\{ \dot{y} \right\} + \left[{}^R \dot{v}'_{G_K,y} \right]\left\{ y \right\} + \left[{}^R \tilde{\omega}'_K \right]\left\{ {}^R v'_{G_K} \right\} \end{aligned}} \quad \text{(all components in body } K\text{)}$$

Using Eq. (32), the ***time derivatives*** of the ***partial velocities*** of the ***mass-centers*** can be written as follows.

$$\boxed{\begin{aligned} \left[{}^R \dot{v}'_{G_K,y} \right] &= \left[{}^J R_K \right]\left[{}^R \dot{v}'_{O_K,y} \right] + \left[{}^J \dot{R}_K \right]\left[{}^R v'_{O_K,y} \right] - \left[\tilde{r}'_K \right]\left[{}^R \dot{\omega}'_{K,y} \right] \\ &= \left[{}^J R_K \right]\left[{}^R \dot{v}'_{O_K,y} \right] + \left[{}^J \tilde{\omega}'_K \right]^T \left[{}^J R_K \right]\left[{}^R v'_{O_K,y} \right] - \left[\tilde{r}'_K \right]\left[{}^R \dot{\omega}'_{K,y} \right] \end{aligned}} \tag{33}$$

Using Eq. (28), the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the bodies can be calculated as follows.

$$\left[{}^R \dot{v}'_{O_K,y} \right] = \left[{}^{\mathscr{L}(J)} R_J \right]\left[{}^R \dot{v}'_{O_J,y} \right] + \left[{}^{\mathscr{L}(J)} \dot{R}_J \right]\left[{}^R v'_{O_J,y} \right] - \left[\dot{\tilde{s}}'_K \right]\left[{}^R \omega'_{J,y} \right] - \left(\left[\tilde{q}'_K \right] + \left[\tilde{s}'_K \right] \right)\left[{}^R \dot{\omega}'_{J,y} \right] + \underbrace{\left[{}^J \dot{v}'_{O_K,y} \right]}_{\text{zero}}$$

$$\boxed{\left[{}^R \dot{v}'_{O_K,y} \right] = \left[{}^{\mathscr{L}(J)} R_J \right]\left[{}^R \dot{v}'_{O_J,y} \right] + \left[{}^{\mathscr{L}(J)} \tilde{\omega}'_J \right]^T \left[{}^{\mathscr{L}(J)} R_J \right]\left[{}^R v'_{O_J,y} \right] - \left[\dot{\tilde{s}}'_K \right]\left[{}^R \omega'_{J,y} \right] - \left(\left[\tilde{q}'_K \right] + \left[\tilde{s}'_K \right] \right)\left[{}^R \dot{\omega}'_{J,y} \right]} \tag{34}$$

This result allows the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the bodies to be calculated in terms of the ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the ***lower bodies***.

Generalized Forces

Let the forces and torques acting on each body of the system be replaced by an ***equivalent force system*** consisting of a single force $\underset{\sim}{F}_K$ acting at the mass-center G_K and a single moment $\underset{\sim}{M}_K$. Then the ***generalized forces*** for the system can be calculated as

$$\boxed{F_{y_i} = \sum_{K=1}^{N} \left(\left(\underset{\sim}{F}_K \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i} \right) + \left(\underset{\sim}{M}_K \cdot \frac{\partial\, {}^R \underset{\sim}{\omega}_K}{\partial y_i} \right) \right)} \tag{35}$$

or, in component form, the column vector of generalized forces is

$$\boxed{\left\{ F_y \right\}_{6N \times 1} = \sum_{K=1}^{N} \left(\left[{}^R v'_{G_K,y} \right]^T \left\{ F'_K \right\} + \left[{}^R \omega'_{K,y} \right]^T \left\{ M'_K \right\} \right)} \tag{36}$$

where $\left\{F_K'\right\}$ represents the body K components of the force-vector $\underset{\sim}{F}_K$ and $\left\{M_K'\right\}$ represents the body K components of the moment-vector $\underset{\sim}{M}_K$.

<u>Equations of Motion of the Unconstrained System</u>

Assuming all "$6N$" of the generalized speeds are ***independent***, Kane's equations of motion for the multibody system can be written as

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times \underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}= F_{y_i}}\qquad (i=1,\ldots,6N)\qquad (37)$$

Here, the ***generalized forces*** on the right side of the equation are the entries of the generalized force column vector of Eq. (36). The terms on the left side of the equation can be written as follows.

1. $\underset{\sim}{a}_{G_K} \;\rightarrow\; \left\{{}^R a'_{G_K}\right\}=\left[{}^R v'_{G_K,y}\right]\{\dot y\}+\left[{}^R \dot v'_{G_K,y}\right]\{y\}+\left[{}^R\tilde\omega'_K\right]\left\{{}^R v'_{G_K}\right\}$

2. ${}^R\underset{\sim}{\alpha}_K \;\rightarrow\; \left\{{}^R\alpha'_K\right\}=\left\{{}^R\dot\omega'_K\right\}=\left[{}^R\omega'_{K,y_1}\right]\{\dot y_1\}+\left[{}^R\dot\omega'_{K,y_1}\right]\{y_1\}$

3. $\displaystyle\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)\rightarrow$

$$\boxed{\begin{aligned}\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left\{{}^R a'_{G_K}\right\}\right)=&\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R v'_{G_K,y}\right]\{\dot y\}\right)\\&+\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R \dot v'_{G_K,y}\right]\{y\}\right)\\&+\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R\tilde\omega'_K\right]\left\{{}^R v'_{G_K}\right\}\right)\end{aligned}}\qquad (38)$$

4. $\displaystyle\sum_{K=1}^{N}\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}\;\rightarrow$

$$\boxed{\sum_{K=1}^{N}\left(\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left\{{}^R\alpha'_K\right\}\right)=\sum_{K=1}^{N}\left(\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left(\left[{}^R\omega'_{K,y_1}\right]\{\dot y_1\}+\left[{}^R\dot\omega'_{K,y_1}\right]\{y_1\}\right)\right)}\qquad (39)$$

5. ${}^R\underset{\sim}{\omega}_K\times \underset{\sim}{H}_{G_K} \;\rightarrow\; \left[{}^R\tilde\omega'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}$ (body-fixed components)

6. $\displaystyle\sum_{K=1}^{N}\left({}^R\underset{\sim}{\omega}_K\times \underset{\sim}{H}_{G_K}\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}\;\rightarrow\;\boxed{\sum_{K=1}^{N}\left[{}^R\omega'_{K,y}\right]^T\left[{}^R\tilde\omega'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}}\qquad (40)$

Substituting from Eqs. (36) and (38)-(40) into Eq. (37) gives

$$\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R v'_{G_K,y}\right]+\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\omega'_{K,y}\right]\right)\{\dot{y}\}$$

$$=\sum_{K=1}^{N}\left(\left[{}^R v'_{G_K,y}\right]^T\left\{F'_K\right\}+\left[{}^R\omega'_{K,y}\right]^T\left\{M'_K\right\}\right)-\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R\dot{v}'_{G_K,y}\right]\{y\}\right)$$

$$-\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R\tilde{\omega}'_K\right]\left\{{}^R v'_{G_K}\right\}\right)-\sum_{K=1}^{N}\left(\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\dot{\omega}'_{K,y}\right]\{y\}\right)$$

$$-\sum_{K=1}^{N}\left[{}^R\omega'_{K,y}\right]^T\left[{}^R\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}$$

The above result can be written in the final matrix form

$$\boxed{[A]\{\dot{y}\}=\{f\}} \qquad ([A] \text{ is called the "\textbf{\textit{generalized mass matrix}}"}) \tag{41}$$

Here,

$$\boxed{[A]=\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R v'_{G_K,y}\right]+\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\omega'_{K,y}\right]\right)} \tag{42}$$

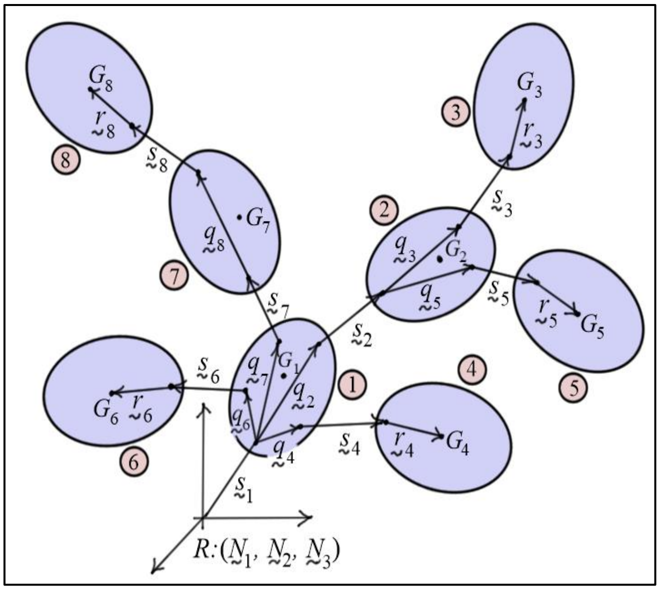
$$\begin{aligned}\{f\}=&\sum_{K=1}^{N}\left(\left[{}^R v'_{G_K,y}\right]^T\left\{F'_K\right\}+\left[{}^R\omega'_{K,y}\right]^T\left\{M'_K\right\}\right)-\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R\dot{v}'_{G_K,y}\right]\{y\}\right)\\&-\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R\tilde{\omega}'_K\right]\left\{{}^R v'_{G_K}\right\}\right)-\sum_{K=1}^{N}\left(\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\dot{\omega}'_{K,y}\right]\{y\}\right)\\&-\sum_{K=1}^{N}\left[{}^R\omega'_{K,y}\right]^T\left[{}^R\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}\end{aligned} \tag{43}$$

Eq. (41) represents "$6N$" ***first-order, ordinary differential equations*** for the "$12N$" variables defined by the system state vectors $\{x\}$ and $\{y\}$ of Eq. (1). To form a ***complete set*** of ***differential equations***, Eq. (41) must be supplemented the set of "$6N$" ***first-order, kinematical differential equations*** defined in Eq. (1).

$$\boxed{\{\dot{x}\}_{6N\times 1}=\{y\}_{6N\times 1}} \tag{44}$$

Example Eight-Body System

As an ***example*** of how to create the ***kinematical quantities*** required to generate the equations of motion of a multibody system, consider the eight-body system shown in the diagram. The equations provided above are used below to generate the angular velocities and partial angular velocities of the bodies and their time derivatives and the velocities and partial velocities of the mass centers of the bodies and their time derivatives. For convenience, note that body 1 is the system's reference body, and the bodies are given increasing numbers moving outward from that body.



Multibody System with Eight Bodies

Eq. (11) states that the ***angular velocity*** of a body K can be written in terms of the ***angular velocity*** of its ***adjacent***, ***lower body*** J.

$$\left\{{}^R\omega'_K\right\} = \left[{}^J R_K\right]\left\{{}^R\omega'_J\right\} + \left\{\hat{\omega}'_K\right\} \qquad \left\{\hat{\omega}'_K\right\} \triangleq \left\{{}^J\omega'_K\right\}$$ (recall the "prime" indicates body-frame components)

This result allows the ***angular velocities*** to be calculated ***recursively***, starting with the ***reference body*** of the system.

$$\left\{{}^R\omega'_1\right\} = \left\{\hat{\omega}'_1\right\}$$ (components in body *1*)

$$\left\{{}^R\omega'_2\right\} = \left[{}^1 R_2\right]\left\{{}^R\omega'_1\right\} + \left\{\hat{\omega}'_2\right\}$$ (components in body *2*)

$$\left\{{}^R\omega'_3\right\} = \left[{}^2 R_3\right]\left\{{}^R\omega'_2\right\} + \left\{\hat{\omega}'_3\right\}$$ (components in body *3*)

$$\left\{{}^R\omega'_4\right\} = \left[{}^1 R_4\right]\left\{{}^R\omega'_1\right\} + \left\{\hat{\omega}'_4\right\}$$ (components in body *4*)

$$\left\{{}^R\omega'_5\right\} = \left[{}^2 R_5\right]\left\{{}^R\omega'_2\right\} + \left\{\hat{\omega}'_5\right\}$$ (components in body *5*)

$$\left\{{}^R\omega'_6\right\} = \left[{}^1 R_6\right]\left\{{}^R\omega'_1\right\} + \left\{\hat{\omega}'_6\right\}$$ (components in body *6*)

$$\left\{{}^R\omega'_7\right\} = \left[{}^1 R_7\right]\left\{{}^R\omega'_1\right\} + \left\{\hat{\omega}'_7\right\}$$ (components in body *7*)

$$\left\{{}^R\omega'_8\right\} = \left[{}^7 R_8\right]\left\{{}^R\omega'_7\right\} + \left\{\hat{\omega}'_8\right\}$$ (components in body *8*)

Eqs. (15)-(19) show how to build the ***partial angular velocity matrix*** of a body K using the ***partial angular velocity matrix*** of its ***adjacent***, ***lower body*** J. Specifically,

$$\left[{}^R\omega'_{K,y_1}\right]_{3\times 3N} = \left[{}^J R_K\right]\left[{}^R\omega'_{J,y_1}\right] + \left[\hat{\omega}'_{K,y_1}\right] \quad \text{and} \quad \left[{}^R\omega'_{K,y_2}\right]_{3\times 3N} = [0]_{3\times 3N}$$

Recall that the "prime" indicates body-frame components. Also, using Eqs. (16) and (45), note that

$$\boxed{\left[\hat{\omega}'_{K,y_1}\right]_{3\times 3N} = \Big[\underset{1}{[0]},[0],\ldots,\underset{K-1}{[0]},\underset{K}{[\hat{P}_K^{AV}]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]}\Big]} \quad \text{and} \quad \boxed{\left[\hat{\omega}'_{K,y_2}\right]_{3\times 3N} = [0]_{3\times 3N}} \quad (K=1,\ldots,N)$$

with

$$\boxed{\left[\hat{P}_K^{AV}\right]_{3\times 3} \triangleq \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}} \quad (K=1,\ldots,N)$$

Applying these results to the ***example eight-body system*** gives the partial velocity matrices associated with state vector y_1. All partitions are 3×3 matrices. As noted above the partial velocity matrices associated with state vector y_2 are all zero.

$$\boxed{\left[{}^R\omega'_{1,y_1}\right]_{3\times 24} = \left[\hat{\omega}'_{1,y_1}\right] = \Big[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\Big]}$$

$$\boxed{\left[{}^R\omega'_{2,y_1}\right]_{3\times 24} = \left[{}^1R_2\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\hat{\omega}'_{2,y_1}\right] = \Big[\left[{}^1R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\Big]}$$

$$\boxed{\begin{aligned}\left[{}^R\omega'_{3,y_1}\right]_{3\times 24} &= \left[{}^2R_3\right]\left[{}^R\omega'_{2,y_1}\right] + \left[\hat{\omega}'_{3,y_1}\right] \\ &= \Big[\left[{}^2R_3\right]\left[{}^1R_2\right]\left[\hat{P}_1^{AV}\right],\left[{}^2R_3\right]\left[\hat{P}_2^{AV}\right],\left[\hat{P}_3^{AV}\right],[0],[0],[0],[0],[0]\Big]\end{aligned}}$$

$$\boxed{\left[{}^R\omega'_{4,y_1}\right]_{3\times 24} = \left[{}^1R_4\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\hat{\omega}'_{4,y_1}\right] = \Big[\left[{}^1R_4\right]\left[\hat{P}_1^{AV}\right],[0],[0],\left[\hat{P}_4^{AV}\right],[0],[0],[0],[0]\Big]}$$

$$\boxed{\begin{aligned}\left[{}^R\omega'_{5,y_1}\right]_{3\times 24} &= \left[{}^2R_5\right]\left[{}^R\omega'_{2,y_1}\right] + \left[\hat{\omega}'_{5,y_1}\right] \\ &= \Big[\left[{}^2R_5\right]\left[{}^1R_2\right]\left[\hat{P}_1^{AV}\right],\left[{}^2R_5\right]\left[\hat{P}_2^{AV}\right],[0],[0],\left[\hat{P}_5^{AV}\right],[0],[0],[0]\Big]\end{aligned}}$$

$$\boxed{\left[{}^R\omega'_{6,y_1}\right]_{3\times 24} = \left[{}^1R_6\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\hat{\omega}'_{6,y_1}\right] = \Big[\left[{}^1R_6\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],\left[\hat{P}_6^{AV}\right],[0],[0]\Big]}$$

$$\boxed{\left[{}^R\omega'_{7,y_1}\right]_{3\times 24} = \left[{}^1R_7\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\hat{\omega}'_{7,y_1}\right] = \Big[\left[{}^1R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[\hat{P}_7^{AV}\right],[0]\Big]}$$

$$\boxed{\begin{aligned}\left[{}^R\omega'_{8,y_1}\right]_{3\times 24} &= \left[{}^7R_8\right]\left[{}^R\omega'_{7,y_1}\right] + \left[\hat{\omega}'_{8,y_1}\right] \\ &= \Big[\left[{}^7R_8\right]\left[{}^1R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[{}^7R_8\right]\left[\hat{P}_7^{AV}\right],\left[\hat{P}_8^{AV}\right]\Big]\end{aligned}}$$

Although these results were generated using Eq. (15), they are the ***same*** as those found by simply ***differentiating*** the expressions for the angular velocities.

Eq. (20) states that the ***time-derivative*** of the ***partial angular velocity matrix*** of body K can be written in terms the ***time-derivative*** of the ***partial angular velocity matrix*** of its ***adjacent***, ***lower body*** J as follows.

$$\boxed{\left[{}^R\dot{\omega}'_{K,y_1}\right]_{3\times 3N} = \left[{}^JR_K\right]\left[{}^R\dot{\omega}'_{J,y_1}\right] + \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\left[{}^R\omega'_{J,y_1}\right] + \left[\dot{\hat{\omega}}'_{K,y_1}\right]} \quad \text{and} \quad \boxed{\left[{}^R\dot{\omega}'_{K,y_2}\right]_{3\times 3N} = [0]} \quad (K=1,\ldots,N)$$

When applying the first of these equations to the eight-body system, it is helpful to note that

$$\left[\dot{\hat{P}}_K^{AV}\right]=\begin{bmatrix} -\left(S_{K2}C_{K3}\dot{\hat{\theta}}_{K2}+C_{K2}S_{K3}\dot{\hat{\theta}}_{K3}\right) & C_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ \left(S_{K2}S_{K3}\dot{\hat{\theta}}_{K2}-C_{K2}C_{K3}\dot{\hat{\theta}}_{K3}\right) & -S_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ C_{K2}\dot{\hat{\theta}}_{K2} & 0 & 0 \end{bmatrix} \quad (K=1,\ldots,N)$$

The time derivatives of the partial angular velocities of the ***eight-body system*** with respect to $\{y_1\}$ can now be written as follows. The time derivatives of the partial angular velocities of the ***eight-body system*** with respect to $\{y_2\}$ are all zero.

$$\left[{}^R\dot{\omega}'_{1,y_1}\right]_{3\times 24}=\left[\dot{\hat{\omega}}'_{K,y_1}\right]=\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{2,y_1}\right]_{3\times 24}&=\left[{}^1R_2\right]\left[{}^R\dot{\omega}'_{1,y_1}\right]+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^R\omega'_{1,y_1}\right]+\left[\dot{\hat{\omega}}'_{2,y_1}\right]\\ &=\left[{}^1R_2\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[[0],\left[\dot{\hat{P}}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{3,y_1}\right]_{3\times 48}&=\left[{}^2R_3\right]\left[{}^R\dot{\omega}'_{2,y_1}\right]+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[{}^R\omega'_{2,y_1}\right]+\left[\dot{\hat{\omega}}'_{3,y_1}\right]\\ &=\left[{}^2R_3\right]\left[{}^1R_2\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[{}^2R_3\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[{}^2R_3\right]\left[[0],\left[\dot{\hat{P}}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[\left[{}^1R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[[0],[0],\left[\dot{\hat{P}}_3^{AV}\right],[0],[0],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{4,y_1}\right]_{3\times 24}&=\left[{}^1R_4\right]\left[{}^R\dot{\omega}'_{1,y_1}\right]+\left[{}^1\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]\left[{}^R\omega'_{1,y_1}\right]+\left[\dot{\hat{\omega}}'_{4,y_1}\right]\\ &=\left[{}^1R_4\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[{}^1\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\ &\quad+\left[[0],[0],[0],\left[\dot{\hat{P}}_4^{AV}\right],[0],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{5,y_1}\right]_{3\times 24} &= \left[{}^2R_5\right]\left[{}^R\dot{\omega}'_{2,y_1}\right] + \left[{}^2\tilde{\omega}'_5\right]^T\left[{}^2R_5\right]\left[{}^R\omega'_{2,y_1}\right] + \left[\dot{\hat{\omega}}'_{5,y_1}\right]\\
&= \left[{}^2R_5\right]\left[{}^1R_2\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^2R_5\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^2R_5\right]\left[[0],\left[\dot{\hat{P}}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^2\tilde{\omega}'_5\right]^T\left[{}^2R_5\right]\left[\left[{}^1R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[[0],[0],[0],[0],\left[\dot{\hat{P}}_5^{AV}\right],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{6,y_1}\right]_{3\times 24} &= \left[{}^1R_6\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\dot{\hat{\omega}}'_{6,y_1}\right]\\
&= \left[{}^1R_6\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^1\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[[0],[0],[0],[0],[0],\left[\dot{\hat{P}}_6^{AV}\right],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{7,y_1}\right]_{3\times 24} &= \left[{}^1R_7\right]\left[{}^R\dot{\omega}'_{1,y_1}\right] + \left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[{}^R\omega'_{1,y_1}\right] + \left[\dot{\hat{\omega}}'_{7,y_1}\right]\\
&= \left[{}^1R_7\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[[0],[0],[0],[0],[0],[0],\left[\dot{\hat{P}}_7^{AV}\right],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R\dot{\omega}'_{8,y_1}\right]_{3\times 24} &= \left[{}^7R_8\right]\left[{}^R\dot{\omega}'_{7,y_1}\right] + \left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]\left[{}^R\omega'_{7,y_1}\right] + \left[\dot{\hat{\omega}}'_{8,y_1}\right]\\
&= \left[{}^7R_8\right]\left[{}^1R_7\right]\left[\left[\dot{\hat{P}}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^7R_8\right]\left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0]\right]\\
&\quad + \left[{}^7R_8\right]\left[[0],[0],[0],[0],[0],[0],\left[\dot{\hat{P}}_7^{AV}\right],\ [0]\right]\\
&\quad + \left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]\left[\left[{}^1R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[\hat{P}_7^{AV}\right],[0]\right]\\
&\quad + \left[[0],[0],[0],[0],[0],[0],[0],\left[\dot{\hat{P}}_8^{AV}\right]\right]\end{aligned}$$

These results are the ***same*** as those found by simply ***differentiating*** the expressions for the partial angular velocities.

Eq. (25) states that the ***velocity*** of the ***origin*** of body K can be written in terms of the ***velocity*** of the ***origin*** of its ***adjacent***, ***lower body*** as follows.

$$\boxed{\left\{{}^{R}v'_{O_K}\right\}=\left[{}^{\mathscr{L}(J)}R_J\right]\left\{{}^{R}v'_{O_J}\right\}+\left\{\dot{s}'_K\right\}+\left[{}^{R}\tilde{\omega}'_J\right]\left\{\left\{q'_K\right\}+\left\{s'_K\right\}\right\}}$$

Eq. (27) states that the ***velocity*** of the ***mass-center*** of body K can be written in terms of the ***velocity*** of the ***origin*** of the body as follows.

$$\boxed{\left\{{}^{R}v'_{G_K}\right\}=\left[{}^{J}R_K\right]\left\{{}^{R}v'_{O_K}\right\}+\left[{}^{R}\tilde{\omega}'_K\right]\left\{r'_K\right\}}$$

Applying these equations to the ***example eight-body system*** gives the following.

$$\boxed{\left\{{}^{R}v'_{O_1}\right\}=\left\{\dot{s}'_1\right\}}$$

$$\boxed{\left\{{}^{R}v'_{G_1}\right\}=\left[R_1\right]\left\{{}^{R}v'_{O_1}\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{r'_1\right\}=\left[R_1\right]\left\{\dot{s}'_1\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{r'_1\right\}}$$

$$\boxed{\left\{{}^{R}v'_{O_2}\right\}=\left[R_1\right]\left\{{}^{R}v'_{O_1}\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}=\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{G_2}\right\}&=\left[{}^{1}R_2\right]\left\{{}^{R}v'_{O_2}\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{r'_2\right\}\\&=\left[{}^{1}R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}\right)+\left[{}^{R}\tilde{\omega}'_2\right]\left\{r'_2\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{O_3}\right\}&=\left[{}^{1}R_2\right]\left\{{}^{R}v'_{O_2}\right\}+\left\{\dot{s}'_3\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\}+\left\{s'_3\right\}\right\}\\&=\left[{}^{1}R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}\right)+\left\{\dot{s}'_3\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\}+\left\{s'_3\right\}\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{G_3}\right\}&=\left[{}^{2}R_3\right]\left\{{}^{R}v'_{O_3}\right\}+\left[{}^{R}\tilde{\omega}'_3\right]\left\{r'_3\right\}\\&=\left[{}^{2}R_3\right]\left(\left[{}^{1}R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}\right)+\left\{\dot{s}'_3\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_3\right\}+\left\{s'_3\right\}\right\}\right)+\left[{}^{R}\tilde{\omega}'_3\right]\left\{r'_3\right\}\end{aligned}}$$

$$\boxed{\left\{{}^{R}v'_{O_4}\right\}=\left[R_1\right]\left\{{}^{R}v'_{O_1}\right\}+\left\{\dot{s}'_4\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\}+\left\{s'_4\right\}\right\}=\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_4\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\}+\left\{s'_4\right\}\right\}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{G_4}\right\}&=\left[{}^{1}R_4\right]\left\{{}^{R}v'_{O_4}\right\}+\left[{}^{R}\tilde{\omega}'_4\right]\left\{r'_4\right\}\\&=\left[{}^{1}R_4\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_4\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_4\right\}+\left\{s'_4\right\}\right\}\right)+\left[{}^{R}\tilde{\omega}'_4\right]\left\{r'_4\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{O_5}\right\}&=\left[{}^{1}R_2\right]\left\{{}^{R}v'_{O_2}\right\}+\left\{\dot{s}'_5\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\}+\left\{s'_5\right\}\right\}\\&=\left[{}^{1}R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}\right)+\left\{\dot{s}'_5\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\}+\left\{s'_5\right\}\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{G_5}\right\}&=\left[{}^{2}R_5\right]\left\{{}^{R}v'_{O_5}\right\}+\left[{}^{R}\tilde{\omega}'_5\right]\left\{r'_5\right\}\\&=\left[{}^{2}R_5\right]\left(\left[{}^{1}R_2\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_2\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_2\right\}+\left\{s'_2\right\}\right\}\right)+\left\{\dot{s}'_5\right\}+\left[{}^{R}\tilde{\omega}'_2\right]\left\{\left\{q'_5\right\}+\left\{s'_5\right\}\right\}\right)+\left[{}^{R}\tilde{\omega}'_5\right]\left\{r'_5\right\}\end{aligned}}$$

$$\boxed{\left\{{}^{R}v'_{O_6}\right\}=\left[R_1\right]\left\{{}^{R}v'_{O_1}\right\}+\left\{\dot{s}'_6\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\}+\left\{s'_6\right\}\right\}=\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_6\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\}+\left\{s'_6\right\}\right\}}$$

$$\boxed{\begin{aligned}\left\{{}^{R}v'_{G_6}\right\}&=\left[{}^{1}R_6\right]\left\{{}^{R}v'_{O_6}\right\}+\left[{}^{R}\tilde{\omega}'_6\right]\left\{r'_6\right\}\\&=\left[{}^{1}R_6\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\}+\left\{\dot{s}'_6\right\}+\left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_6\right\}+\left\{s'_6\right\}\right\}\right)+\left[{}^{R}\tilde{\omega}'_6\right]\left\{r'_6\right\}\end{aligned}}$$

$$\left\{{}^{R}v'_{O_7}\right\} = \left[R_1\right]\left\{{}^{R}v'_{O_1}\right\} + \left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\} = \left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}$$

$$\begin{aligned} \left\{{}^{R}v'_{G_7}\right\} &= \left[{}^{1}R_7\right]\left\{{}^{R}v'_{O_7}\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left\{r'_7\right\} \\ &= \left[{}^{1}R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left[{}^{R}\tilde{\omega}'_7\right]\left\{r'_7\right\} \end{aligned}$$

$$\begin{aligned} \left\{{}^{R}v'_{O_8}\right\} &= \left[{}^{1}R_7\right]\left\{{}^{R}v'_{O_7}\right\} + \left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\} \\ &= \left[{}^{1}R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\} \end{aligned}$$

$$\begin{aligned} \left\{{}^{R}v'_{G_8}\right\} &= \left[{}^{7}R_8\right]\left\{{}^{R}v'_{O_8}\right\} + \left[{}^{R}\tilde{\omega}'_8\right]\left\{r'_8\right\} \\ &= \left[{}^{7}R_8\right]\left(\left[{}^{1}R_7\right]\left(\left[R_1\right]\left\{\dot{s}'_1\right\} + \left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left\{\left\{q'_7\right\} + \left\{s'_7\right\}\right\}\right) + \left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left\{\left\{q'_8\right\} + \left\{s'_8\right\}\right\}\right) + \left[{}^{R}\tilde{\omega}'_8\right]\left\{r'_8\right\} \end{aligned}$$

Eqs. (28)-(32) state that the ***partial velocity*** of the ***origin*** of body K can be written in terms of the ***partial velocity*** of the ***origin*** of its ***adjacent***, ***lower body*** J. The partial velocities of the origins of the bodies can be found as follows.

$$\left[{}^{R}v'_{O_K,y}\right] = \left[{}^{\mathcal{L}(J)}R_J\right]\left[{}^{R}v'_{O_J,y}\right] - \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{J,y}\right] + \left[{}^{J}v'_{O_K,y}\right]$$

where the last term $\left[{}^{J}v'_{O_K,y}\right]$ can be ***partitioned*** and ***defined*** as

$$\left[{}^{J}v'_{O_K,y}\right]_{3\times 6N} = \left[\left[{}^{J}v'_{O_K,y_1}\right]_{3\times 3N}\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N}\right] = \left[\left[0\right]_{3\times 3N}\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N}\right]$$

with $\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N}$ defined in a partitioned form as follows.

$$\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N} = \left[\underset{1}{[0]},\ldots,\underset{K-1}{[0]},\underset{K}{[I]},\underset{K+1}{[0]},\ldots,\underset{N}{[0]}\right]$$

These results can be used to find the partial velocities of the mass centers of the bodies as follows.

$$\left[{}^{R}v'_{G_K,y}\right] = \left[{}^{J}R_K\right]\left[{}^{R}v'_{O_K,y}\right] - \left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]$$

Applying these equations to the ***example eight-body*** system gives the following.

$$\left[{}^{R}v'_{O_1,y}\right]_{3\times 48} = \left[{}^{R}v'_{O_1,y}\right] = \left[\left[0\right]_{3\times 24}\left[{}^{J}v'_{O_1,y_2}\right]_{3\times 24}\right] = \left[\left[0\right]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$\begin{aligned} \left[{}^{R}v'_{G_1,y}\right]_{3\times 48} &= \left[R_1\right]\left[{}^{R}v'_{O_1,y}\right] - \left[\tilde{r}'_1\right]\left[{}^{R}\omega'_{1,y}\right] \\ &= \left[R_1\right]\left[\left[0\right]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\ &\quad - \left[\tilde{r}'_1\right]\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],\left[0\right]_{3\times 24}\right] \end{aligned}$$

$$\left[{}^R v'_{O_2,y}\right]_{3\times 48} = [R_1]\left[{}^R v'_{O_1,y}\right] - \left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[{}^R \omega'_{1,y}\right] + \left[{}^1 v'_{O_2,y}\right]$$
$$= [R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$

$$\left[{}^R v'_{G_2,y}\right]_{3\times 48} = \left[{}^1 R_2\right]\left[{}^R v'_{O_2,y}\right] - \left[\tilde{r}'_2\right]\left[{}^R \omega'_{2,y}\right]$$
$$= \left[{}^1 R_2\right][R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^1 R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[{}^1 R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[\tilde{r}'_2\right]\left[\left[{}^1 R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$

$$\left[{}^R v'_{O_3,y}\right]_{3\times 48} = \left[{}^1 R_2\right]\left[{}^R v'_{O_2,y}\right] - \left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^R \omega'_{2,y}\right] + \left[{}^2 v'_{O_3,y}\right]$$
$$= \left[{}^1 R_2\right][R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^1 R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[{}^1 R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[\left[{}^1 R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right]$$

$$\left[{}^R v'_{G_3,y}\right] = \left[{}^2 R_3\right]\left[{}^R v'_{O_3,y}\right] - \left[\tilde{r}'_3\right]\left[{}^R \omega'_{3,y}\right]$$
$$= \left[{}^2 R_3\right]\left[{}^1 R_2\right][R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^2 R_3\right]\left[{}^1 R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[{}^2 R_3\right]\left[{}^1 R_2\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^2 R_3\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[\left[{}^1 R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[{}^2 R_3\right]\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right]$$
$$-\left[\tilde{r}'_3\right]\left[\left[{}^2 R_3\right]\left[{}^1 R_2\right]\left[\hat{P}_1^{AV}\right],\left[{}^2 R_3\right]\left[\hat{P}_2^{AV}\right],\left[\hat{P}_3^{AV}\right],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$

$$\left[{}^R v'_{O_4,y}\right] = [R_1]\left[{}^R v'_{O_1,y}\right] - \left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^R \omega'_{1,y}\right] + \left[{}^1 v'_{O_4,y}\right]$$
$$= [R_1]\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]$$
$$+\left[[0]_{3\times 24},[0],[0],[0],[I],[0],[0],[0],[0]\right]$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_4,y}\right] &= \left[{}^{1}R_4\right]\left[{}^{R}v'_{O_4,y}\right] - \left[\tilde{r}'_4\right]\left[{}^{R}\omega'_{4,y}\right] \\
&= \left[{}^{1}R_4\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_4\right]\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[{}^{1}R_4\right]\left[[0]_{3\times24},[0],[0],[0],[I],[0],[0],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_4\right]\left[\left[{}^{1}R_4\right]\left[\hat{P}_1^{AV}\right],[0],[0],\left[\hat{P}_4^{AV}\right],[0],[0],[0],[0],[0]_{3\times24}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{O_5,y}\right] &= \left[{}^{1}R_2\right]\left[{}^{R}v'_{O_2,y}\right] - \left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[{}^{R}\omega'_{2,y}\right] + \left[{}^{2}v'_{O_5,y}\right] \\
&= \left[{}^{1}R_2\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[{}^{1}R_2\right]\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[\left[{}^{1}R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[[0]_{3\times24},[0],[0],[0],[0],[I],[0],[0],[0]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_5,y}\right] &= \left[{}^{2}R_5\right]\left[{}^{R}v'_{O_5,y}\right] - \left[\tilde{r}'_5\right]\left[{}^{R}\omega'_{5,y}\right] \\
&= \left[{}^{2}R_5\right]\left[{}^{1}R_2\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{2}R_5\right]\left[{}^{1}R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[{}^{2}R_5\right]\left[{}^{1}R_2\right]\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{2}R_5\right]\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[\left[{}^{1}R_2\right]\left[\hat{P}_1^{AV}\right],\left[\hat{P}_2^{AV}\right],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[{}^{2}R_5\right]\left[[0]_{3\times24},[0],[0],[0],[0],[I],[0],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_5\right]\left[\left[{}^{2}R_5\right]\left[{}^{1}R_2\right]\left[\hat{P}_1^{AV}\right],\left[{}^{2}R_5\right]\left[\hat{P}_2^{AV}\right],[0],[0],\left[\hat{P}_5^{AV}\right],[0],[0],[0],[0]_{3\times24}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{O_6,y}\right]_{3\times48} &= \left[R_1\right]\left[{}^{R}v'_{O_1,y}\right] - \left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[{}^{R}\omega'_{1,y}\right] + \left[{}^{1}v'_{O_6,y}\right] \\
&= \left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[[0]_{3\times24},[0],[0],[0],[0],[0],[I],[0],[0]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v'_{G_6,y}\right]_{3\times48} &= \left[{}^{1}R_6\right]\left[{}^{R}v'_{O_6,y}\right] - \left[\tilde{r}'_6\right]\left[{}^{R}\omega'_{6,y}\right] \\
&= \left[{}^{1}R_6\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[{}^{1}R_6\right]\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right] \\
&\quad + \left[{}^{1}R_6\right]\left[[0]_{3\times24},[0],[0],[0],[0],[0],[I],[0],[0]\right] \\
&\quad - \left[\tilde{r}'_6\right]\left[\left[{}^{1}R_6\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],\left[\hat{P}_6^{AV}\right],[0],[0],[0]_{3\times24}\right]
\end{aligned}
$$

$$\left[{}^{R}v'_{O_7,y}\right]_{3\times48} = \left[R_1\right]\left[{}^{R}v'_{O_1,y}\right] - \left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[{}^{R}\omega'_{1,y}\right] + \left[{}^{1}v'_{O_7,y}\right]$$
$$= \left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$
$$+\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[I],[0]\right]$$

$$\left[{}^{R}v'_{G_7,y}\right] = \left[{}^{1}R_7\right]\left[{}^{R}v'_{O_7,y}\right] - \left[\tilde{r}'_7\right]\left[{}^{R}\omega'_{7,y}\right]$$
$$= \left[{}^{1}R_7\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^{1}R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$
$$+\left[{}^{1}R_7\right]\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[I],[0]\right]$$
$$-\left[\tilde{r}'_7\right]\left[\left[{}^{1}R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[\hat{P}_7^{AV}\right],[0],[0]_{3\times24}\right]$$

$$\left[{}^{R}v'_{O_8,y}\right] = \left[{}^{1}R_7\right]\left[{}^{R}v'_{O_7,y}\right] - \left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[{}^{R}\omega'_{7,y}\right] + \left[{}^{7}v'_{O_8,y}\right]$$
$$= \left[{}^{1}R_7\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^{1}R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$
$$+\left[{}^{1}R_7\right]\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[I],[0]\right]$$
$$-\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[\left[{}^{1}R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[\hat{P}_7^{AV}\right],[0],[0]_{3\times24}\right]$$
$$+\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[0],[I]\right]$$

$$\left[{}^{R}v'_{G_8,y}\right] = \left[{}^{7}R_8\right]\left[{}^{R}v'_{O_8,y}\right] - \left[\tilde{r}'_8\right]\left[{}^{R}\omega'_{8,y}\right]$$
$$= \left[{}^{7}R_8\right]\left[{}^{1}R_7\right]\left[R_1\right]\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$
$$-\left[{}^{7}R_8\right]\left[{}^{1}R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$
$$+\left[{}^{7}R_8\right]\left[{}^{1}R_7\right]\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[I],[0]\right]$$
$$-\left[{}^{7}R_8\right]\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[\left[{}^{1}R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[\hat{P}_7^{AV}\right],[0],[0]_{3\times24}\right]$$
$$+\left[{}^{7}R_8\right]\left[[0]_{3\times24},[0],[0],[0],[0],[0],[0],[0],[I]\right]$$
$$-\left[\tilde{r}'_8\right]\left[\left[{}^{7}R_8\right]\left[{}^{1}R_7\right]\left[\hat{P}_1^{AV}\right],[0],[0],[0],[0],[0],\left[{}^{7}R_8\right]\left[\hat{P}_7^{AV}\right],\left[\hat{P}_8^{AV}\right],[0]_{3\times24}\right]$$

Finally, Eqs. (33) and (34) state the ***time derivative*** of the ***partial velocity*** of the ***mass-center*** of body K can be written in terms of the ***time derivative*** of the ***partial velocity*** of the ***origin*** of body K, and that the ***time derivative*** of the ***partial velocity*** of the ***origin*** of body K can be written in terms of the ***time derivative*** of the ***partial velocity*** of the ***origin*** of its ***adjacent***, ***lower body***. Specifically,

$$\left[{}^{R}\dot{v}'_{O_K,y}\right] = \left[{}^{\mathcal{L}(J)}R_J\right]\left[{}^{R}\dot{v}'_{O_J,y}\right] + \left[{}^{\mathcal{L}(J)}\tilde{\omega}'_J\right]^T\left[{}^{\mathcal{L}(J)}R_J\right]\left[{}^{R}v'_{O_J,y}\right] - \left[\dot{\tilde{s}}'_K\right]\left[{}^{R}\omega'_{J,y}\right] - \left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^{R}\dot{\omega}'_{J,y}\right]$$

and

$$\left[{}^R\dot{v}'_{G_K,y}\right]=\left[{}^JR_K\right]\left[{}^R\dot{v}'_{O_K,y}\right]+\left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\left[{}^Rv'_{O_K,y}\right]-\left[\tilde{r}'_K\right]\left[{}^R\dot{\omega}'_{K,y}\right]$$

Applying these equations to the ***example eight-body*** system gives the following results.

$$\left[{}^R\dot{v}'_{O_1,y}\right]_{3\times48}=\left[0\right]_{3\times48}$$

$$\left[{}^R\dot{v}'_{G_1,y}\right]_{3\times48}=\left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right]+\left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right]-\left[\tilde{r}'_1\right]\left[{}^R\dot{\omega}'_{1,y}\right]$$

$$\left[{}^R\dot{v}'_{O_2,y}\right]_{3\times48}=\left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right]+\left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right]-\left[\tilde{\dot{s}}'_2\right]\left[{}^R\omega'_{1,y}\right]-\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]$$

$$\left[{}^R\dot{v}'_{G_2,y}\right]_{3\times48}=\left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right]+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right]-\left[\tilde{r}'_2\right]\left[{}^R\dot{\omega}'_{2,y}\right]$$

$$\left[{}^R\dot{v}'_{O_3,y}\right]_{3\times48}=\left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right]+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right]-\left[\tilde{\dot{s}}'_3\right]\left[{}^R\omega'_{2,y}\right]-\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^R\dot{\omega}'_{2,y}\right]$$

$$\left[{}^R\dot{v}'_{G_3,y}\right]_{3\times48}=\left[{}^2R_3\right]\left[{}^R\dot{v}'_{O_3,y}\right]+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[{}^Rv'_{O_3,y}\right]-\left[\tilde{r}'_3\right]\left[{}^R\dot{\omega}'_{3,y}\right]$$

$$\left[{}^R\dot{v}'_{O_4,y}\right]_{3\times48}=\left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right]+\left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right]-\left[\tilde{\dot{s}}'_4\right]\left[{}^R\omega'_{1,y}\right]-\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]$$

$$\left[{}^R\dot{v}'_{G_4,y}\right]_{3\times48}=\left[{}^1R_4\right]\left[{}^R\dot{v}'_{O_4,y}\right]+\left[{}^1\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]\left[{}^Rv'_{O_4,y}\right]-\left[\tilde{r}'_4\right]\left[{}^R\dot{\omega}'_{4,y}\right]$$

$$\left[{}^R\dot{v}'_{O_5,y}\right]_{3\times48}=\left[{}^1R_2\right]\left[{}^R\dot{v}'_{O_2,y}\right]+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left[{}^Rv'_{O_2,y}\right]-\left[\tilde{\dot{s}}'_5\right]\left[{}^R\omega'_{2,y}\right]-\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[{}^R\dot{\omega}'_{2,y}\right]$$

$$\left[{}^R\dot{v}'_{G_5,y}\right]_{3\times48}=\left[{}^2R_5\right]\left[{}^R\dot{v}'_{O_5,y}\right]+\left[{}^2\tilde{\omega}'_5\right]^T\left[{}^2R_5\right]\left[{}^Rv'_{O_5,y}\right]-\left[\tilde{r}'_5\right]\left[{}^R\dot{\omega}'_{5,y}\right]$$

$$\left[{}^R\dot{v}'_{O_6,y}\right]_{3\times48}=\left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right]+\left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right]-\left[\tilde{\dot{s}}'_6\right]\left[{}^R\omega'_{1,y}\right]-\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]$$

$$\left[{}^R\dot{v}'_{G_6,y}\right]_{3\times48}=\left[{}^1R_6\right]\left[{}^R\dot{v}'_{O_6,y}\right]+\left[{}^1\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]\left[{}^Rv'_{O_6,y}\right]-\left[\tilde{r}'_6\right]\left[{}^R\dot{\omega}'_{6,y}\right]$$

$$\left[{}^R\dot{v}'_{O_7,y}\right]_{3\times48}=\left[R_1\right]\left[{}^R\dot{v}'_{O_1,y}\right]+\left[{}^R\tilde{\omega}'_1\right]^T\left[R_1\right]\left[{}^Rv'_{O_1,y}\right]-\left[\tilde{\dot{s}}'_7\right]\left[{}^R\omega'_{1,y}\right]-\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[{}^R\dot{\omega}'_{1,y}\right]$$

$$\left[{}^R\dot{v}'_{G_7,y}\right]_{3\times48}=\left[{}^1R_7\right]\left[{}^R\dot{v}'_{O_7,y}\right]+\left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[{}^Rv'_{O_7,y}\right]-\left[\tilde{r}'_7\right]\left[{}^R\dot{\omega}'_{7,y}\right]$$

$$\left[{}^R\dot{v}'_{O_8,y}\right]_{3\times48}=\left[{}^1R_7\right]\left[{}^R\dot{v}'_{O_7,y}\right]+\left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\left[{}^Rv'_{O_7,y}\right]-\left[\tilde{\dot{s}}'_8\right]\left[{}^R\omega'_{7,y}\right]-\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[{}^R\dot{\omega}'_{7,y}\right]$$

$$\left[{}^R\dot{v}'_{G_8,y}\right]_{3\times48}=\left[{}^7R_8\right]\left[{}^R\dot{v}'_{O_8,y}\right]+\left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]\left[{}^Rv'_{O_8,y}\right]-\left[\tilde{r}'_8\right]\left[{}^R\dot{\omega}'_{8,y}\right]$$

All terms on the right side of these equations have been previously defined, and because of the length of the resulting expressions, the results from above were not substituted directly into the equations.