Multibody Dynamics
Equations of Motion for Multibody Systems with Inter-Body Joint Constraints

Introduction

These notes describe how the ***equations*** of ***motion*** for multibody systems with ***inter-body joint constraints*** can be formulated from the equations of motion of the ***unconstrained system***. The motion of the ***reference body*** of the system is described relative to an ***inertial frame***, and the motion of all other bodies are described ***relative*** to their ***adjacent***, ***lower bodies***. ***Orientation angles*** are used to define the rotational degrees of freedom, and ***translation variables*** are used to define the translational degrees of freedom.

Comments are made at the end of these notes about the ***convenience*** of using ***multiple reference directions*** for the bodies and about the use of ***Euler parameters*** along with the orientation angles to avoid geometric singularities.

Approaches for the Application of Joint Constraints

It is assumed herein that the equations of motion of a multibody system with inter-body joint constraints are being developed using Lagrange's equations, d'Alembert's Principle, Kane's equations, or some similar procedure. There are many ways to incorporate joint constraints using these procedures. The constraints can be ***implicit*** in the formulation of the equations of motion by choosing a set of generalized coordinates and speeds that specifically represent the degrees of freedom consistent with the joint constraints. In that case, no additional constraint equations are required. The resulting system is unconstrained.

Alternatively, if a ***dependent set*** of generalized coordinates are used that do not specifically represent the degrees of freedom of the joints, a separate set of constraint equations must be incorporated to describe the restrictions the joints place on the generalized coordinates. In this case, the constraint equations are ***explicitly*** included in the development of the equations of motion.

When using a dependent set of generalized coordinates, joints can be created by simply forcing some of the generalized coordinates to have a ***specified value***, such as zero, for example. In that case the equations of motion associated with the ***known coordinates*** are ***eliminated*** from the analysis, and the values of the known coordinates and their derivatives are used in the remaining equations. Once the motion of the rest of the system is determined, the ***eliminated equations*** can then be used to find the constraint forces and torques required to produce the specified values. This is the approach used in these notes.

As discussed in previous notes, the generalized coordinates used to describe multibody systems can be measured relative to a fixed ground (inertial) frame (***absolute coordinates***) or they can be measured relative to other bodies of the system (***relative coordinates***). In these notes, a ***reference body*** is identified, labelled body 1 (or B_1), and its generalized coordinates are measured relative to an ***inertial frame***. The coordinates associated with all other bodies are measured ***relative*** to ***adjacent***, ***lower bodies***. The lower body is one body closer to the reference body. Without loss of generality, it is assumed that bodies have ***increasing body numbers*** moving outward along the branches, so lower bodies are also ***lower-numbered bodies***. A body connection array (presented in previous notes) is used to identify the lower bodies. ***Interconnecting joints*** are formed between bodies and their adjacent, lower bodies.

Joints can range from totally free joints to totally locked (or rigid) joints. The ***table*** below lists a set of ***common*** and ***relatively simple joints***. Modeling of more complex joints is limited only by the analyst's ability to write an ***accurate set*** of ***constraint equations***. Note, however, that complex joints can often be modeled as a series of simpler joints. For example, a universal joint can be modeled as a two degree-of-freedom rotational joint, or it can be modeled as two single degree-of-freedom joints by adding a third body between the two adjoining bodies.

Analysts should find a ***balance*** between the ***complexity*** of the ***joint*** (and the resulting constraint equations required to maintain it) and the ***number*** of ***extra bodies*** needed to simplify the joints. The use of joints that are too complex can lead to erroneous results if the constraint equations associated with the joint do not accurately represent the physics of the modelled joint.

As indicated in the table, ***spherical*** (or ball-and-socket) ***joints*** eliminate the translational motion between two bodies while allowing all three degrees of rotational freedom; ***universal joints*** eliminate the translational motion between two bodies while allowing two degrees of rotational freedom; ***cylindrical joints*** eliminate two translational and two rotational degrees-of-freedom allowing translation and rotation along a single axis; ***hinge joints*** eliminate all translational degrees-of-freedom and two rotational degrees-of-freedom allowing only rotation about a single axis; and ***prismatic joints*** eliminate all three rotational degrees-of-freedom and two of the three translational degrees-of-freedom allowing only translation along a single axis. A discussion of how to incorporate joints such as these into the equations of motion is presented below.

Joint Type	***Degrees of Freedom***
Rigid	0
Hinge (revolute)	1 (1 rotation)
Spherical (ball & socket)	3 (3 rotation)
Two-Angle (universal)	2 (2 rotation)
Prismatic (slider)	1 (1 translation)
Cylindrical	2 (1 translation, 1 rotation)
Free	6 (3 translation, 3 rotation)

Equations of Motion of the Unconstrained System

To create joints in a multibody system by eliminating unnecessary generalized coordinates associated with relative motion between the bodies, first formulate the equations of motion of the system with no restrictions between the bodies. Then, the equations can be sorted into ***two sets***, one set associated with the ***unknown***

coordinates, and one set associated with the ***known coordinates***. This will be accomplished herein using relative orientation angles, relative translation variables, and their derivatives as the generalized coordinates and speeds.

Orientation Angles: (for a system of N bodies)

Orientation angles $\hat{\theta}_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure the ***orientations*** of the bodies ***relative*** to their ***adjacent, lower bodies***. For illustration purposes, the orientation angles $\hat{\theta}_{Ki}$ $(i=1,2,3)$ are chosen to be the 1-2-3 body-fixed orientation angles of body K relative to its lower body $\mathscr{L}(K)$. The angles $\hat{\theta}_{1i}$ $(i=1,2,3)$ are used to measure the orientation of the reference body (body 1) relative to the inertial frame.

Translation variables:

The translation variables s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure ***displacements*** of the bodies ***relative*** to their ***adjacent, lower bodies***. These variables represent the ***lower-body-frame components*** of the relative translation vectors of the bodies $\left(\underset{\sim}{s}_K\ (K=1,\ldots,N)\right)$. The vector $\underset{\sim}{s}_1$ measures the translation of the reference body relative to the inertial frame.

As described, there are "$6N$" generalized coordinates, $\hat{\theta}_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$ and s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ for the unconstrained system. Using these coordinates, the system state vectors can be defined as follows.

$$\left\{\hat{\theta}\right\}_{3N\times1}=[\hat{\theta}_{11},\hat{\theta}_{12},\hat{\theta}_{13},\ldots,\underbrace{\hat{\theta}_{K1},\hat{\theta}_{K2},\hat{\theta}_{K3}}_{\left\{\hat{\theta}_K\right\}^T},\ldots,\hat{\theta}_{N1},\hat{\theta}_{N2},\hat{\theta}_{N3}]^T$$

$$\left\{s'\right\}_{3N\times1}=[s'_{11},s'_{12},s'_{13},\ldots,\underbrace{s'_{K1},s'_{K2},s'_{K3}}_{\left\{s'_K\right\}^T},\ldots,s'_{N1},s'_{N2},s'_{N3}]^T$$

and

$$\left\{x\right\}_{6N\times1}=\begin{Bmatrix}\left\{x_1\right\}\\ \left\{x_2\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{\hat{\theta}\right\}_{3N\times1}\\ \left\{s'\right\}_{3N\times1}\end{Bmatrix}\qquad\qquad \left\{y\right\}_{6N\times1}=\begin{Bmatrix}\left\{y_1\right\}\\ \left\{y_2\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{\dot{\hat{\theta}}\right\}_{3N\times1}\\ \left\{\dot{s}'\right\}_{3N\times1}\end{Bmatrix} \tag{1}$$

Equations of Motion

It was shown in previous notes that the ***equations*** of ***motion*** of the ***unconstrained system*** can be written using ***Kane's equations*** in the matrix form,

$$\boxed{[A]\left\{\dot{y}\right\}=\left\{f\right\}}\qquad ([A] \text{ is called the "\textbf{\textit{generalized mass matrix}}"}) \tag{2}$$

Here,

$$\boxed{[A]=\sum_{K=1}^{N}\left(m_K\left[{}^R v'_{G_K,y}\right]^T\left[{}^R v'_{G_K,y}\right]+\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\omega'_{K,y}\right]\right)} \tag{3}$$

$$\boxed{\begin{aligned}\{f\} = &\sum_{K=1}^{N}\left(\left[{}^{R}v'_{G_K,y}\right]^{T}\left\{F'_K\right\}+\left[{}^{R}\omega'_{K,y}\right]^{T}\left\{M'_K\right\}\right)-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\dot{v}'_{G_K,y}\right]\{y\}\right)\\ &-\sum_{K=1}^{N}\left(m_K\left[{}^{R}v'_{G_K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_K\right]\left\{{}^{R}v'_{G_K}\right\}\right)-\sum_{K=1}^{N}\left(\left[{}^{R}\omega'_{K,y}\right]^{T}\left[I'_{G_K}\right]\left[{}^{R}\dot{\omega}'_{K,y}\right]\{y\}\right)\\ &-\sum_{K=1}^{N}\left[{}^{R}\omega'_{K,y}\right]^{T}\left[{}^{R}\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^{R}\omega'_K\right\}\end{aligned}} \qquad (4)$$

The individual terms in Eqs. (3) and (4) are defined as follows.

G_K	- mass-center of body K
m_K	- mass of body K
$\left[I'_{G_K}\right]_{3\times3}$	- inertia matrix of body K for G_K (body-frame components)
$\left[{}^{R}\omega'_{K,y}\right]_{3\times6N}$	- partial angular velocity matrix of body K (body-frame components)
$\left[{}^{R}\dot{\omega}'_{K,y}\right]_{3\times6N}$	- time derivative of $\left[{}^{R}\omega'_{K,y}\right]_{3\times6N}$
$\left[{}^{R}v'_{G_K,y}\right]_{3\times6N}$	- partial velocity matrix for G_K (body-frame components)
$\left[{}^{R}\dot{v}'_{G_K,y}\right]_{3\times6N}$	- time derivative of $\left[{}^{R}v'_{G_K,y}\right]_{3\times6N}$
$\left\{{}^{R}\omega'_K\right\}_{3\times1}$	- body-frame components of ${}^{R}\underset{\sim}{\omega}_K$
$\left\{{}^{R}v'_{G_K}\right\}_{3\times1}$	- body-frame components of ${}^{R}\underset{\sim}{v}_{G_K}$
$\left[{}^{R}\tilde{\omega}'_K\right]_{3\times3}$	- skew-symmetric matrix containing body-frame components of ${}^{R}\underset{\sim}{\omega}_K$
$\left\{F'_K\right\}_{3\times1}$	- body-frame components of resultant force acting at G_K
$\left\{M'_K\right\}_{3\times1}$	- body-frame components of resultant torque acting on body K

Eq. (2) represents "$6N$" ***first-order, ordinary differential equations*** for the "$12N$" variables defined by the $\{x\}$ and $\{y\}$ vectors of Eq. (1). To form a ***complete set*** of ***differential equations***, Eq. (2) is supplemented with the following set of "$6N$" ***first-order, kinematical differential equations***.

$$\boxed{\{\dot{x}\}_{6N\times1}=\{y\}_{6N\times1}} \qquad (5)$$

Details of the matrices needed in Eqs. (3) and (4) are provided in previous notes (#40) entitled "Equations of Motion for Unconstrained Systems Using Relative Coordinates with Orientation Angles".

Elimination of Unnecessary Generalized Coordinates to Create Inter-Body Joints

<u>Kane's Equations with Lagrange Multipliers</u>

Eq. (2) was generated using Kane's equations without constraints. To incorporate ***joint constraints*** into the equations of motion, first a set of constraint equations must be written in the form

$$\boxed{\sum_{i=1}^{6N} b_{ji}y_i + b_{j0} \;=\; 0} \qquad (j=1,\ldots,m) \qquad (6)$$

Using the coefficients $b_{jk}\ \left(j=1,\ldots m;\ k=1,6N\right)$, Kane's equations can be written as follows.

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}=\ F_{y_i}+\sum_{j=1}^{m}\lambda_j b_{ji}}\quad (i=1,\ldots,6N) \tag{7}$$

with

$$\boxed{F_{y_i}=\sum_{K=1}^{N}\left(\left(\underset{\sim}{F}_K\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\left(\underset{\sim}{M}_K\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}\right)\right)}\qquad (i=1,\ldots,6N) \tag{8}$$

Here, $\lambda_j\ \left(j=1,\ldots,m\right)$ are a set of ***Lagrange multipliers*** associated with the constraints. Except for the multiplier terms, Eqs. (7) are identical to those used to create Eq. (2).

If the k^{th} generalized speed is known to be ***zero***, Eq. (6) becomes

$$\boxed{\Big[\underset{1}{0}\ ,\ \underset{2}{0}\ ,0\,,0\,,\ldots,\ \underset{k-1}{0}\ ,\ \underset{k}{1}\ ,\ \underset{k+1}{0}\ ,0\,,0\,,\ldots,\ \underset{6N}{0}\Big]\{y\}_{6N\times 1}=0} \tag{9}$$

Using this result in Eq. (7) gives

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}=\ F_{y_i}}\qquad (i=1,\ldots,k-1) \tag{10}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}=\ F_{y_i}+\lambda}\qquad (i=k) \tag{11}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}=\ F_{y_i}}\qquad (i=k+1,\ldots,6N) \tag{12}$$

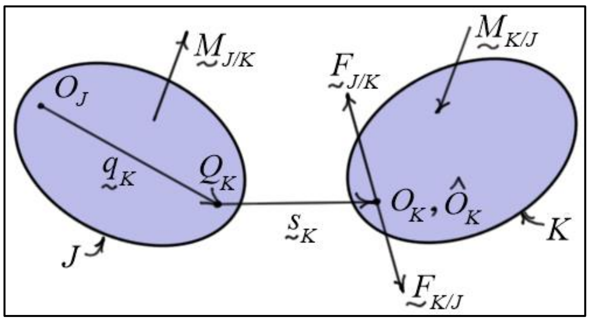
Eqs. (10) and (12) are supplemented with the kinematical Eqs. (5) to solve for the $6N-1$ generalized coordinate $x_i\ (i=1,2,\ldots,k-1,k+1,\ldots,6N)$ and the $6N-1$ generalized speeds $y_i\ (i=1,2,\ldots,k-1,k+1,\ldots,6N)$. After solving these equations, the value of the Lagrange multiplier λ can be found ***algebraically*** using Eq. (11). Given $y_k=0$, x_k must be constant. These values are used in the solution process.

Note in the above analysis the coefficients $b_{ji}\ (j=1,\ldots,m;\ i=0,\ldots,6N)$ can be ***functions*** of the ***generalized coordinates*** and ***time***. Including a time-dependent coefficient b_0 in Eq. (9), for example, allows values of y_k and x_k to be ***time-varying***. In that case, the values of y_k and x_k must be updated during the solution of the equations of motion to be consistent with the time-varying constraint equation.

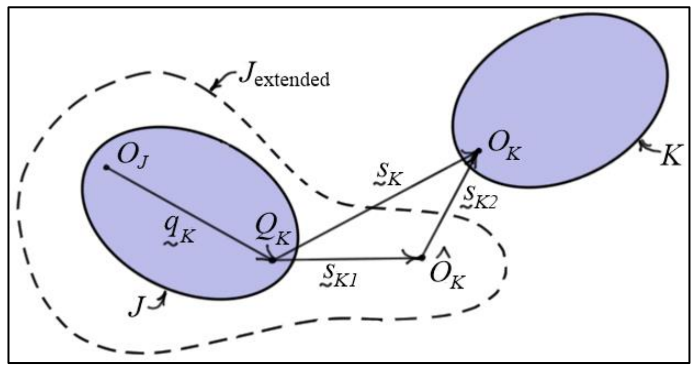
Constraint Relaxation – Constraint Forces and Torques Between Adjoining Bodies

As presented above, joints that constrain the motion between adjoining bodies can be formed using constraint equations along with the equations of motion of the unconstrained system. Solution of the equations produces ***time-dependent values*** for the ***generalized coordinates*** and ***speeds*** as well as the ***Lagrange multipliers*** associated with the constraints. The Lagrange multipliers are related to the ***constraint forces*** and ***torques*** required to maintain the constraints.

To see how the Lagrange multipliers are related to the constraint forces and torques consider two ***typical adjoining bodies*** *J* and *K* as shown in the diagrams below. Here, *J* is the ***lower-numbered*** body of *K*, so using the body-connection array, $J = \mathscr{L}(K)$. Let the ***constraint force*** and ***torque*** applied ***to*** body *K* ***by*** body *J* be $\underset{\sim}{F}_{K/J}$ applied at O_K and $\underset{\sim}{M}_{K/J}$. Similarly, let the ***constraint force*** and ***torque*** applied ***to*** body *J* ***by*** body *K* be $\underset{\sim}{F}_{J/K}$ applied at $\hat{O}_K$ and $\underset{\sim}{M}_{J/K}$. Here, $\hat{O}_K$ is a point on body J_{extended}, that is, it is a point on body *J* that coincides with point O_K on body *K*.



Constraint Forces and Torques between Adjoining Bodies



Artificial Separation of Two Adjoining Bodies

By Newton's third law, the forces and torques the bodies exert on each other are equal in magnitude and opposite in direction. Hence, the contribution of these constraint forces and torques to the generalized forces of the system can be calculated as follows.

$$
\begin{aligned}
F_{y_i}^c &= \left(\underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K}}{\partial y_i} \right) + \left(\underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \right) + \left(\underset{\sim}{F}_{J/K} \cdot \frac{\partial \underset{\sim}{v}_{\hat{O}_K}}{\partial y_i} \right) + \left(\underset{\sim}{M}_{J/K} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_J}{\partial y_i} \right) \\
&= \left(\underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K}}{\partial y_i} \right) + \left(\underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \right) - \left(\underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{\hat{O}_K}}{\partial y_i} \right) - \left(\underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_J}{\partial y_i} \right) \\
&= \underset{\sim}{F}_{K/J} \cdot \left(\frac{\partial \underset{\sim}{v}_{O_K}}{\partial y_i} - \frac{\partial \underset{\sim}{v}_{\hat{O}_K}}{\partial y_i} \right) + \underset{\sim}{M}_{K/J} \cdot \left(\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} - \frac{\partial\, {}^R\underset{\sim}{\omega}_J}{\partial y_i} \right) \\
&= \underset{\sim}{F}_{K/J} \cdot \frac{\partial}{\partial y_i}\left(\underset{\sim}{v}_{O_K} - \underset{\sim}{v}_{\hat{O}_K} \right) + \underset{\sim}{M}_{K/J} \cdot \frac{\partial}{\partial y_i}\left({}^R\underset{\sim}{\omega}_K - {}^R\underset{\sim}{\omega}_J \right)
\end{aligned}
$$

$$
\Rightarrow \boxed{F_{y_i}^c = \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i}} \qquad (13)
$$

Eq. (13) indicates the ***contributions*** of the ***constraint forces*** depends on the ***velocity*** of O_K ***relative*** to $\hat{O}_K$, and the ***contribution*** of the ***constraint torques*** depends on the ***angular velocity*** of body *K* ***relative*** to body *J*. Hence, the contribution of the joint constraint force and torque as defined in Eq. (13) is ***zero*** for any y_i ***not associated*** with body *K*.

The modified equations of motion can now be written as follows. (Note Eq. (14) does not apply for $K=1$ and Eq. (18) does not apply for $K=N$.)

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}=F_{y_i}}\qquad \left(i=1,\ldots,3K-3\right) \tag{14}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}=F_{y_i}+F_{y_i}^c}\qquad \left(i=3K-2,\ 3K-1,\ 3K\right) \tag{15}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}=F_{y_i}}\qquad \left(i=3K+1,\ldots,3(N+K)-3\right) \tag{16}$$

$$\boxed{\begin{array}{l}\displaystyle\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}\\ \qquad =F_{y_i}+F_{y_i}^c\end{array}}\qquad \left(\begin{array}{l}i=3(N+K)-2,\\ \qquad 3(N+K)-1,\ 3(N+K)\end{array}\right) \tag{17}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot {}^R\underset{\sim}{\alpha}_K\right)+\left({}^R\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}=F_{y_i}}\qquad \left(i=3(N+K)+1,\ldots,6N\right) \tag{18}$$

The terms on the right side of these equations are defined by Eqs. (8) and (13).

<u>Partial Relative Angular Velocities</u>

The form of the components of ${}^J\underset{\sim}{\omega}_K$ the angular velocity of body *K* relative to body *J* depends on the ***choice*** of orientation angles. For a 1-2-3 sequence of rotations, it is shown in previous notes that the body *K* components of the relative angular velocity appearing in Eq. (13) are related to the derivatives of the orientation angles of body *K* as follows.

$$\boxed{\left\{{}^J\omega_{(K)}\right\}_{3\times1}=\left\{\hat{\omega}'_K\right\}_{3\times1}=\begin{Bmatrix}\hat{\omega}'_{K1}\\ \hat{\omega}'_{K2}\\ \hat{\omega}'_{K3}\end{Bmatrix}=\begin{bmatrix}C_{K2}C_{K3} & S_{K3} & 0\\ -C_{K2}S_{K3} & C_{K3} & 0\\ S_{K2} & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\hat{\theta}}_{K1}\\ \dot{\hat{\theta}}_{K2}\\ \dot{\hat{\theta}}_{K3}\end{Bmatrix}} \tag{19}$$

Parentheses surround the body *K* indicator on the left side of the equation to indicate these are body *K* components. The body *K* components of the partial angular velocity vectors can be written as follows.

$$\boxed{\frac{\partial\,\hat{\underset{\sim}{\omega}}_K}{\partial y_{3K-2}}=C_{K2}C_{K3}\,\underset{\sim}{n}_{K1}-C_{K2}S_{K3}\,\underset{\sim}{n}_{K2}+S_{K2}\,\underset{\sim}{n}_{K3}}\quad \boxed{\frac{\partial\,\hat{\underset{\sim}{\omega}}_K}{\partial y_{3K-1}}=S_{K3}\,\underset{\sim}{n}_{K1}+C_{K3}\,\underset{\sim}{n}_{K2}}\quad \boxed{\frac{\partial\,\hat{\underset{\sim}{\omega}}_K}{\partial y_{3K}}=\underset{\sim}{n}_{K3}} \tag{20}$$

$$\boxed{\frac{\partial\,\hat{\underset{\sim}{\omega}}_K}{\partial y_i}=\underset{\sim}{0}}\qquad \left(i\neq 3K-2,3K-1,3K\right) \tag{21}$$

For a 1-2-3 rotation sequence, it can also be shown that the body *J* components of the relative angular velocity of Eq. (13) are related to the derivatives of the orientation angles of body *K* as follows.

$$\left\{{}^{(J)}\omega_K\right\}_{3\times1} = \left\{\hat{\omega}_K\right\}_{3\times1} = \begin{Bmatrix} \hat{\omega}_{K1} \\ \hat{\omega}_{K2} \\ \hat{\omega}_{K3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \tag{22}$$

Note the parentheses around the body *J* label indicating these are body *J* components. Using Eq. (22), the partial angular velocity vectors can be expressed in the body *J* reference frame as follows.

$$\frac{\partial \hat{\underset{\sim}{\omega}}_K}{\partial y_{3K-2}} = \underset{\sim}{n}_{J1} \qquad \frac{\partial \hat{\underset{\sim}{\omega}}_K}{\partial y_{3K-1}} = C_{K1}\,\underset{\sim}{n}_{J2} + S_{K1}\,\underset{\sim}{n}_{J3} \qquad \frac{\partial \hat{\underset{\sim}{\omega}}_K}{\partial y_{3K}} = S_{K2}\,\underset{\sim}{n}_{J1} - S_{K1}C_{K2}\,\underset{\sim}{n}_{J2} + C_{K1}C_{K2}\,\underset{\sim}{n}_{J3} \tag{23}$$

$$\frac{\partial \hat{\underset{\sim}{\omega}}_K}{\partial y_i} = \underset{\sim}{0} \qquad \left(i \neq 3K-2,\, 3K-1,\, 3K\right) \tag{24}$$

<u>Partial Relative Velocities</u>

To find the partial velocity of Eq. (13), the velocity of O_K relative to $\hat{O}_K$ is calculated as follows.

$$\begin{aligned} \underset{\sim}{v}_{O_K/\hat{O}_K} &= \frac{{}^R d}{dt}\left(\underset{\sim}{s}_{K2}\right) = \frac{{}^R d}{dt}\left(\underset{\sim}{s}_K - \underset{\sim}{s}_{K1}\right) = \frac{{}^R d}{dt}\left(\underset{\sim}{s}_K\right) - \frac{{}^R d}{dt}\left(\underset{\sim}{s}_{K1}\right) = \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \underset{\sim}{s}_K - {}^R\underset{\sim}{\omega}_J \times \underset{\sim}{s}_{K1} \\ &= \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{s}_K - \underset{\sim}{s}_{K1}\right) \\ &= \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{s}_{K2}\right) \end{aligned}$$

Because O_K coincides with $\hat{O}_K$, the above result can now be simplified by setting $\underset{\sim}{s}_{K2} = 0$ to give

$$\underset{\sim}{v}_{O_K/\hat{O}_K} = \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) \tag{25}$$

Given that the components of the displacement vector $\underset{\sim}{s}_K$ are expressed in body *J*, the relative velocity components of Eq. (25) are also elements of the vector of generalized speeds. Specifically,

$$\left\{v^{(J)}_{O_K/\hat{O}_K}\right\}_{3\times1} = \left\{\dot{s}'_K\right\}_{3\times1} \tag{26}$$

The results expressed in Eqs. (20), (21), and (26) can now be used to calculate the constraint force and torque contributions as given in Eq. (13). The contributions can be divided into one of three cases:

Case	Description
1	y_i is ***not*** one of $\dot{\hat{\theta}}_{Kj}$ $(j=1,2,3)$ and is ***not*** one of $\dot{s}'_{Kj}$ $(j=1,2,3)$
2	y_i is one of $\dot{\hat{\theta}}_{Kj}$ $(j=1,2,3)$
3	y_i is one of $\dot{s}'_{Kj}$ $(j=1,2,3)$

Details of each of the ***three cases*** are shown below. The force components are expressed in the lower-body (J) frame. The moment components are expressed in both the body K frame and the lower-body J frame. ***Parentheses*** enclose the body label in which the components are resolved.

Case 1: $\left\langle\; i \neq 3K-2,\; 3K-1, \text{ or } 3K \quad \text{and} \quad i \neq 3(N+K)-2,\; 3(N+K)-1, \text{ or } 3(N+K) \;\right\rangle$

$y_i \neq \dot{\hat{\theta}}_{Kj}$ $(j=1,2,3)$ and $y_i \neq \dot{s}'_{Kj}$ $(j=1,2,3)$

$$\boxed{\frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \underset{\sim}{0}}$$

$$\Rightarrow \boxed{F^c_{y_i} = \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = 0} \tag{27}$$

Case 2: $\left\langle\; i = 3K-3+j \quad \text{with} \quad (j=1,2,3) \;\right\rangle$ … ***using body*** *K* ***components***

$y_i = \dot{\hat{\theta}}_{Kj}$ $(j=1,2, \text{ or } 3)$

$$\boxed{\frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} = \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial \dot{\hat{\theta}}_{Kj}} = \underset{\sim}{0}}$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K-2}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K1}} = C_{K2}C_{K3}\,\underset{\sim}{n}_{K1} - C_{K2}S_{K3}\,\underset{\sim}{n}_{K2} + S_{K2}\,\underset{\sim}{n}_{K3}}$$

$$\Rightarrow \boxed{\begin{aligned} F^c_{y_i} &= \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} \\ &= \underset{\sim}{M}_{K/J} \cdot \left(C_{K2}C_{K3}\,\underset{\sim}{n}_{K1} - C_{K2}S_{K3}\,\underset{\sim}{n}_{K2} + S_{K2}\,\underset{\sim}{n}_{K3}\right) \\ &= \left(C_{K2}C_{K3}\right)M_1^{(K)/J} - \left(C_{K2}S_{K3}\right)M_2^{(K)/J} + \left(S_{K2}\right)M_3^{(K)/J} \end{aligned}} \quad (i = 3K-2) \tag{28}$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K-1}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K2}} = S_{K3}\,\underset{\sim}{n}_{K1} + C_{K3}\,\underset{\sim}{n}_{K2}}$$

$$\Rightarrow \boxed{\begin{aligned} F_{y_i}^c &== \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} \\ &= \underset{\sim}{M}_{K/J} \cdot \left(S_{K3}\, \underset{\sim}{n}_{K1} + C_{K3}\, \underset{\sim}{n}_{K2} \right) \\ &= \left(S_{K3}\right) M_1^{(K)/J} + \left(C_{K3}\right) M_2^{(K)/J} \end{aligned}} \quad \left(i = 3K-1\right) \qquad (29)$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K3}} = \underset{\sim}{n}_{K3}}$$

$$\Rightarrow \boxed{F_{y_i}^c = \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \underset{\sim}{M}_{K/J} \cdot \underset{\sim}{n}_{K3} = M_3^{(K)/J}} \quad \left(i = 3K\right) \qquad (30)$$

Case 2: $\left\langle\ i = 3K - 3 + j \quad \text{with} \quad \left(j = 1,2,3 \right) \right\rangle$ … ***using body J components***

$y_i = \dot{\hat{\theta}}_{Kj}\ (j = 1, 2, \text{ or } 3)$

$$\boxed{\frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} = \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial \dot{\hat{\theta}}_{Kj}} = \underset{\sim}{0}}$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K-2}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K1}} = \underset{\sim}{n}_{J1}}$$

$$\Rightarrow \boxed{F_{y_i}^c = \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \underset{\sim}{M}_{K/J} \cdot \underset{\sim}{n}_{J1} = M_1^{K/(J)}} \quad \left(i = 3K-2\right) \qquad (31)$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K-1}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K2}} = C_{K1}\, \underset{\sim}{n}_{J2} + S_{K1}\, \underset{\sim}{n}_{J3}}$$

$$\Rightarrow \boxed{\begin{aligned} F_{y_i}^c &= \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \underset{\sim}{M}_{K/J} \cdot \left(C_{K1}\, \underset{\sim}{n}_{J2} + S_{K1}\, \underset{\sim}{n}_{J3} \right) \\ &= \left(C_{K1}\right) M_2^{K/(J)} + \left(S_{K1}\right) M_3^{K/(J)} \end{aligned}} \quad \left(i = 3K-1\right) \qquad (32)$$

$$\boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_{3K}} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{\hat{\theta}}_{K3}} = S_{K2}\, \underset{\sim}{n}_{J1} - S_{K1} C_{K2}\, \underset{\sim}{n}_{J2} + C_{K1} C_{K2}\, \underset{\sim}{n}_{J3}}$$

$$\Rightarrow \boxed{\begin{aligned} F_{y_i}^c &= \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} \\ &= \underset{\sim}{M}_{K/J} \cdot \left(S_{K2}\, \underset{\sim}{n}_{J1} - S_{K1} C_{K2}\, \underset{\sim}{n}_{J2} + C_{K1} C_{K2}\, \underset{\sim}{n}_{J3} \right) \\ &= \left(S_{K2}\right) M_1^{K/(J)} - \left(S_{K1} C_{K2}\right) M_2^{K/(J)} + \left(C_{K1} C_{K2}\right) M_3^{K/(J)} \end{aligned}} \quad \left(i = 3K\right) \qquad (33)$$

Case 3: $\left\langle\, i = 3\left(N+K\right)-3+j \quad \text{with} \quad \left(\, j=1,2,3\,\right) \right\rangle$

$y_i = \dot{s}'_{Kj} \quad \left(j=1,2, \text{ or } 3\right)$

$$\boxed{\frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} = \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial \dot{s}'_{Kj}} = \underset{\sim}{e}_{Jj}} \quad \boxed{\frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial \dot{s}'_{Kj}} = \underset{\sim}{0}} \quad \left(j=1,2, \text{ or } 3\right)$$

$$\Rightarrow \boxed{F^c_{y_i} = \underset{\sim}{F}_{K/J} \cdot \frac{\partial \underset{\sim}{v}_{O_K/\hat{O}_K}}{\partial y_i} + \underset{\sim}{M}_{K/J} \cdot \frac{\partial\, {}^J\underset{\sim}{\omega}_K}{\partial y_i} = \underset{\sim}{F}_{K/J} \cdot \underset{\sim}{e}_{Jj} = F_j^{K/(J)}} \text{ (body } J \text{ frame components)} \qquad (34)$$

<u>Spherical (ball-and-socket) Joints</u>

Spherical joints ***eliminate all translational degrees*** of ***freedom*** between adjoining bodies, but they ***allow all*** three ***degrees*** of ***rotational freedom***. Using the notation from the figures above, this can be accomplished between bodies *J* and *K*, for example, by forcing Q_K to be coincident with O_K, that is by setting

$$\boxed{\underset{\sim}{s}_K = \dot{\underset{\sim}{s}}_K = \ddot{\underset{\sim}{s}}_K \equiv 0} \qquad (35)$$

In terms of the generalized coordinates, this means that s'_{Ki} $(i=1,2,3)$ and $\dot{s}'_{Ki}$ $(i=1,2,3)$ are ***known coordinates*** and speeds with a value of ***zero***.

$$\boxed{\begin{Bmatrix} s'_{K1} \\ s'_{K2} \\ s'_{K3} \end{Bmatrix} = \begin{Bmatrix} \dot{s}'_{K1} \\ \dot{s}'_{K2} \\ \dot{s}'_{K3} \end{Bmatrix} \equiv \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}} \qquad (36)$$

These constraints are maintained throughout the motion of the system with three components of constraint force between bodies *J* and *K*. Using Lagrange multipliers (one for each of the three constraints), the equations of motion can be written as follows.

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i} \right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} = F_{y_i}} \quad \left(i=1,\ldots,3(N+K)-3\right) \qquad (37)$$

$$\boxed{\begin{aligned}&\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i} \right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \\ &\quad = F_{y_i} + \lambda_j\end{aligned}} \quad \left(i=3(N+K)-3+j\right)\ \left(j=1,2,3\right) \qquad (38)$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i} \right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} = F_{y_i}} \quad \left(i=3(N+K)+1,\ldots,6N\right) \qquad (39)$$

Using the constraint force components, the equations of motion can be written as

$$\boxed{\sum_{K=1}^{N}\left(m_K\, \underset{\sim}{a}_{G_K} \cdot \frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i} \right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} = F_{y_i}} \quad \left(i=1,\ldots,3(N+K)-3\right) \qquad (40)$$

$$\boxed{\begin{aligned}&\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}\\&\quad=\;F_{y_i}+F_j^{K/(J)}\end{aligned}}\quad \big(i=3(N+K)-3+j\big)\;\big(j=1,2,3\big)\tag{41}$$

$$\boxed{\sum_{K=1}^{N}\left(m_K\,\underset{\sim}{a}_{G_K}\cdot\frac{\partial \underset{\sim}{v}_{G_K}}{\partial y_i}\right)+\sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K}\cdot{}^{R}\underset{\sim}{\alpha}_K\right)+\left({}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{H}_{G_K}\right)\right]\cdot\frac{\partial\,{}^{R}\underset{\sim}{\omega}_K}{\partial y_i}=\;F_{y_i}}\quad \big(i=3(N+K)+1,\ldots,6N\big)\tag{42}$$

Comparing Eqs. (38) and (41), it is clear the ***Lagrange multipliers*** are simply the ***body*** *J* ***components*** of the ***constraint force***. So, Eqs. (40) and (42) can be solved for the unknown coordinates and speeds (with the known coordinates and speeds set to zero). Then, Eqs. (41) can be used to solve ***algebraically*** for each of the constraint force components.

Using the above approach, every spherical joint in the system will reduce the order of the square coefficient matrix $\left[A\right]$ of Eq. (2) by three. The rows and columns of $\left[A\right]$ and the rows of $\left\{\dot{y}\right\}$ and $\left\{f\right\}$ associated with $\dot{s}'_{Ki}$ $(i=1,2,3)$ are eliminated to solve for the motion of the system. The reduced matrix equation represents Eqs. (40) and (42).

<u>Universal Joints</u>

Universal joints ***eliminate all translational degrees*** of ***freedom*** (as with the ball and socket joint) and ***one rotational degree*** of ***freedom*** between adjoining bodies. Hence the joint has two rotational degrees of freedom. This can be accomplished between bodies *J* and *K* by, for example, forcing Q_K to be coincident with O_K, and by setting one of the three orientation angles to zero. In terms of the generalized coordinates, Eq. (36) still applies for the translational coordinates and speeds. In addition, ***one*** of the following three rotational constraints apply.

$$\boxed{\hat{\theta}_{Ki}=\dot{\hat{\theta}}_{Ki}=0}\quad \big(i=1,2,\text{ or }3\big)\tag{43}$$

Each of these three constraints produces different results for the associated constraint moment components. But in each case, one of orientation angles and the s'_{Ki} $(i=1,2,3)$ are known and zero. Eliminating the equations associated with these variables will reduce the number of equations of motion by four for each universal joint. The constraint force components of universal joints are found in the same way they are found for spherical joints. The following paragraphs show how to calculate the constraint moment components for each of the possible angle constraints.

Constraint on $\hat{\theta}_{K1}$: $\quad\Rightarrow \boxed{\hat{\theta}_{K1}=\dot{\hat{\theta}}_{K1}=0}\quad\Rightarrow\boxed{S_{K1}=0}\quad\boxed{C_{K1}=1}$

In this case, $y_i\,\big(i=3K-2,\;3(N+K)-2,\;3(N+K)-1,\;3(N+K)\big)$ are ***known variables*** and ***equal*** to ***zero*** so they can be eliminated from the equations of motion. The first Lagrange multiplier is associated with $\hat{\theta}_{K1}$ and the constraint moments, and as above, the last three are the *J* frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body J components,

$$\boxed{\begin{Bmatrix} \lambda_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ (S_{K2})M_1^{K/(J)} - (S_{K1}C_{K2})M_2^{K/(J)} + (C_{K1}C_{K2})M_3^{K/(J)} \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ (S_{K2})M_1^{K/(J)} + (C_{K2})M_3^{K/(J)} \end{Bmatrix}} \tag{44}$$

Solving gives,

$$\boxed{M_1^{K/(J)} = \lambda_1} \qquad \boxed{M_2^{K/(J)} = 0} \qquad \boxed{M_3^{K/(J)} = -(S_{K2}/C_{K2})M_1^{K/(J)} = -(S_{K2}/C_{K2})\lambda_1} \tag{45}$$

Using body K components,

$$\boxed{\begin{Bmatrix} \lambda_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{Bmatrix} = \begin{Bmatrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix}} \tag{46}$$

Solving gives,

$$\boxed{M_3^{(K)/J} = 0} \tag{47}$$

and

$$\begin{aligned} \begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \end{Bmatrix} &= \begin{bmatrix} C_{K2}C_{K3} & -C_{K2}S_{K3} \\ S_{K3} & C_{K3} \end{bmatrix}^{-1} \begin{Bmatrix} \lambda_1 \\ 0 \end{Bmatrix} = \frac{1}{C_{K2}} \begin{bmatrix} C_{K3} & C_{K2}S_{K3} \\ -S_{K3} & C_{K2}C_{K3} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ 0 \end{Bmatrix} \\ &\Rightarrow \boxed{\begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \end{Bmatrix} = \frac{1}{C_{K2}} \begin{Bmatrix} C_{K3} \\ -S_{K3} \end{Bmatrix} \lambda_1} \end{aligned} \tag{48}$$

The K frame components can also be found directly from the J frame components using the transformation matrix $\left[{}^J R_K\right]$. The J frame is rotated into the K frame using a 2-3 rotation sequence, so

$$\boxed{\left[{}^J R_K\right] = \begin{bmatrix} C_{K3} & S_{K3} & 0 \\ -S_{K3} & C_{K3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{K2} & 0 & -S_{K2} \\ 0 & 1 & 0 \\ S_{K2} & 0 & C_{K2} \end{bmatrix} = \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & -S_{K2}C_{K3} \\ -C_{K2}S_{K3} & C_{K3} & S_{K2}S_{K3} \\ S_{K2} & 0 & C_{K2} \end{bmatrix}} \tag{49}$$

Using this transformation, the J frame components can be converted into the K frame components as follows.

$$\begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix} = \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & -S_{K2}C_{K3} \\ -C_{K2}S_{K3} & C_{K3} & S_{K2}S_{K3} \\ S_{K2} & 0 & C_{K2} \end{bmatrix} \begin{Bmatrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ M_3^{K/(J)} \end{Bmatrix} = \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & -S_{K2}C_{K3} \\ -C_{K2}S_{K3} & C_{K3} & S_{K2}S_{K3} \\ S_{K2} & 0 & C_{K2} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ 0 \\ -\left(S_{K2}/C_{K2}\right)\lambda_1 \end{Bmatrix}$$

$$= \begin{Bmatrix} C_{K2}C_{K3} + S_{K2}C_{K3}\left(S_{K2}/C_{K2}\right) \\ -C_{K2}S_{K3} - S_{K2}S_{K3}\left(S_{K2}/C_{K2}\right) \\ S_{K2} - C_{K2}\left(S_{K2}/C_{K2}\right) \end{Bmatrix} \lambda_1$$

$$M_1^{(K)/J} = \left[C_{K2}C_{K3} + S_{K2}C_{K3}\left(S_{K2}/C_{K2}\right)\right]\lambda_1 = C_{K3}\left[\frac{C_{K2}^2 + S_{K2}^2}{C_{K2}}\right]\lambda_1 = \left(C_{K3}/C_{K2}\right)\lambda_1 \tag{50}$$

$$M_2^{(K)/J} = \left[-C_{K2}S_{K3} - S_{K2}S_{K3}\left(S_{K2}/C_{K2}\right)\right]\lambda_1 = -S_{K3}\left[\frac{C_{K2}^2 + S_{K2}^2}{C_{K2}}\right]\lambda_1 = -\left(S_{K3}/C_{K2}\right)\lambda_1 \tag{51}$$

$$M_3^{(K)/J} = \left(S_{K2} - C_{K2}\left(S_{K2}/C_{K2}\right)\right)\lambda_1 = \left[\frac{S_{K2}C_{K2} - S_{K2}C_{K2}}{C_{K2}}\right]\lambda_1 = 0 \tag{52}$$

These are the same results found above in Eqs. (47) and (48).

Constraint on $\hat{\theta}_{K2}$: $\Rightarrow \boxed{\hat{\theta}_{K2} = \dot{\hat{\theta}}_{K2} = 0} \quad \Rightarrow \boxed{S_{K2} = 0} \quad \boxed{C_{K2} = 1}$

In this case, $y_i \left(i = 3K-1,\ 3(N+K)-2,\ 3(N+K)-1,\ 3(N+K)\right)$ are ***known variables*** and ***equal*** to ***zero*** so they can be eliminated from the equations of motion. The first Lagrange multiplier is associated with $\hat{\theta}_{K2}$ and the constraint moments, and the last three are the J frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body J components,

$$\boxed{\begin{Bmatrix} 0 \\ \lambda_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{y_{3K-2}}^c \\ F_{y_{3K-1}}^c \\ F_{y_{3K}}^c \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ \left(C_{K1}\right)M_2^{K/(J)} + \left(S_{K1}\right)M_3^{K/(J)} \\ \left(S_{K2}\right)M_1^{K/(J)} - \left(S_{K1}C_{K2}\right)M_2^{K/(J)} + \left(C_{K1}C_{K2}\right)M_3^{K/(J)} \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ \left(C_{K1}\right)M_2^{K/(J)} + \left(S_{K1}\right)M_3^{K/(J)} \\ -\left(S_{K1}\right)M_2^{K/(J)} + \left(C_{K1}\right)M_3^{K/(J)} \end{Bmatrix}} \tag{53}$$

Solving gives,

$$\boxed{M_1^{K/(J)} = 0}$$

$$\begin{Bmatrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{Bmatrix} = \begin{bmatrix} C_{K1} & S_{K1} \\ -S_{K1} & C_{K1} \end{bmatrix}^{-1} \begin{Bmatrix} \lambda_1 \\ 0 \end{Bmatrix} = \begin{bmatrix} C_{K1} & -S_{K1} \\ S_{K1} & C_{K1} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ 0 \end{Bmatrix} \quad \Rightarrow \boxed{\begin{Bmatrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{Bmatrix} = \begin{Bmatrix} C_{K1} \\ S_{K1} \end{Bmatrix} \lambda_1} \tag{54}$$

Using body K components,

$$\left\{\begin{matrix} 0 \\ \lambda_1 \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K3})M_1^{(K)/J} - (S_{K3})M_2^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} \tag{55}$$

Solving gives,

$$\boxed{M_3^{(K)/J} = 0} \tag{56}$$

and

$$\left\{\begin{matrix} M_1^{(K)/J} \\ M_2^{(K)/J} \end{matrix}\right\} = \begin{bmatrix} C_{K3} & -S_{K3} \\ S_{K3} & C_{K3} \end{bmatrix}^{-1} \left\{\begin{matrix} 0 \\ \lambda_1 \end{matrix}\right\} = \begin{bmatrix} C_{K3} & S_{K3} \\ -S_{K3} & C_{K3} \end{bmatrix} \left\{\begin{matrix} 0 \\ \lambda_1 \end{matrix}\right\} \quad \Rightarrow \boxed{\left\{\begin{matrix} M_1^{(K)/J} \\ M_2^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} S_{K3} \\ C_{K3} \end{matrix}\right\} \lambda_1} \tag{57}$$

It can be easily shown that the results in Eqs. (56) and (57) can also be found using the transformation matrix $\left[{}^J R_K\right]$ which represents a 1-3 rotation sequence.

Constraint on $\hat{\theta}_{K3}$: $\quad \Rightarrow \boxed{\hat{\theta}_{K3} = \dot{\hat{\theta}}_{K3} = 0} \quad \Rightarrow \boxed{S_{K3} = 0} \quad \boxed{C_{K3} = 1}$

In this case, $y_i \left(i = 3K,\ 3(N+K)-2,\ 3(N+K)-1,\ 3(N+K) \right)$ are ***known variables*** and ***equal*** to ***zero*** so they can be eliminated from the equations of motion. The first Lagrange multiplier is associated with $\hat{\theta}_{K3}$ and the constraint moments, and the last three are the *J* frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body *J* components,

$$\left\{\begin{matrix} 0 \\ 0 \\ \lambda_1 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ (S_{K2})M_1^{K/(J)} - (S_{K1}C_{K2})M_2^{K/(J)} + (C_{K1}C_{K2})M_3^{K/(J)} \end{matrix}\right\} \tag{58}$$

Solving gives,

$$\boxed{M_1^{K/(J)} = 0} \tag{59}$$

$$\left\{\begin{matrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{matrix}\right\} = \begin{bmatrix} C_{K1} & S_{K1} \\ -S_{K1}C_{K2} & C_{K1}C_{K2} \end{bmatrix}^{-1} \left\{\begin{matrix} 0 \\ \lambda_1 \end{matrix}\right\} = \frac{1}{C_{K2}} \begin{bmatrix} C_{K1}C_{K2} & -S_{K1} \\ S_{K1}C_{K2} & C_{K1} \end{bmatrix} \left\{\begin{matrix} 0 \\ \lambda_1 \end{matrix}\right\}$$
$$\Rightarrow \boxed{\left\{\begin{matrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{matrix}\right\} = \frac{1}{C_{K2}} \left\{\begin{matrix} -S_{K1} \\ C_{K1} \end{matrix}\right\} \lambda_1} \tag{60}$$

Using body *K* components,

$$\begin{Bmatrix} 0 \\ 0 \\ \lambda_1 \end{Bmatrix} = \begin{Bmatrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{Bmatrix} = \begin{Bmatrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix} = \begin{Bmatrix} (C_{K2})M_1^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix} \tag{61}$$

Solving gives,

$$\boxed{M_3^{(K)/J} = \lambda_1} \quad \boxed{M_2^{(K)/J} = 0} \quad \boxed{M_1^{(K)/J} = -(S_{K2}/C_{K2})M_3^{(K)/J} = -(S_{K2}/C_{K2})\lambda_1} \tag{62}$$

It can be easily shown that the results in Eqs. (62) can also be found using the transformation matrix $\left[{}^J R_K\right]$ which represents a 1-2 rotation sequence.

Hinge Joints

Hinge joints ***eliminate all translational degrees*** of ***freedom*** (as with the ball-and-socket joint) and ***two rotational degrees*** of ***freedom*** between adjoining bodies. Hence the joint has ***one rotational degree*** of ***freedom***. This can be accomplished between bodies *J* and *K* by, for example, forcing Q_K to be coincident with O_K, and by setting ***two*** of the ***three orientation angles*** to ***zero***. In terms of the generalized coordinates, Eq. (36) still applies for the translational coordinates and speeds. In addition, ***two*** of the following three rotational constraints apply.

$$\boxed{\hat{\theta}_{Ki} = \dot{\hat{\theta}}_{Ki} = 0} \quad (i = 1,2,3) \tag{63}$$

Each pair of these constraints produces different results for the associated constraint moment components. But in each case, ***two*** of ***orientation angles*** and the s'_{Ki} $(i = 1,2,3)$ are ***known*** and ***zero***. Eliminating the equations associated with these variables will ***reduce*** the ***number*** of ***equations*** of ***motion*** by ***five*** for each hinge joint. The constraint force components of hinge joints are found in the same way they are found for spherical joints. The following paragraphs show how to calculate the constraint moment components for each possible combination of the angle constraints.

Constraints on $\hat{\theta}_{K2}$ *and* $\hat{\theta}_{K3}$: $\Rightarrow \boxed{\hat{\theta}_{K2} = \dot{\hat{\theta}}_{K2} = \hat{\theta}_{K3} = \dot{\hat{\theta}}_{K3} = 0} \quad \Rightarrow \boxed{S_{K2} = S_{K3} = 0} \quad \Rightarrow \boxed{C_{K2} = C_{K3} = 1}$

In this case, $y_i\ (i = 3K-1, 3K,\ 3(N+K)-2,\ 3(N+K)-1,\ 3(N+K))$ are ***known variables*** and ***equal*** to ***zero***, so they can be eliminated from the equations of motion. The first two Lagrange multipliers are associated with angles $\hat{\theta}_{K2}$ and $\hat{\theta}_{K3}$, and the constraint moments. As above, the last three are the *J* frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body *J* components,

$$\left\{\begin{matrix} 0 \\ \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ (S_{K2})M_1^{K/(J)} - (S_{K1}C_{K2})M_2^{K/(J)} + (C_{K1}C_{K2})M_3^{K/(J)} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ -(S_{K1})M_2^{K/(J)} + (C_{K1})M_3^{K/(J)} \end{matrix}\right\} \quad (64)$$

Solving gives,

$$\boxed{M_1^{K/(J)} = 0} \quad \left\{\begin{matrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{matrix}\right\} = \begin{bmatrix} C_{K1} & S_{K1} \\ -S_{K1} & C_{K1} \end{bmatrix}^{-1} \left\{\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \begin{bmatrix} C_{K1} & -S_{K1} \\ S_{K1} & C_{K1} \end{bmatrix} \left\{\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} \quad (65)$$

Using body K components,

$$\left\{\begin{matrix} 0 \\ \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} \quad (66)$$

The K frame components can also be found directly from the J frame components using the transformation matrix $\left[{}^J R_K\right]$. The J frame is rotated into the K frame using a 1 rotation, so

$$\left[{}^J R_K\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{K1} & S_{K1} \\ 0 & -S_{K1} & C_{K1} \end{bmatrix} \quad (67)$$

$$\left\{\begin{matrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{K1} & S_{K1} \\ 0 & -S_{K1} & C_{K1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ C_{K1} & -S_{K1} \\ S_{K1} & C_{K1} \end{bmatrix} \left\{\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ \lambda_1 \\ \lambda_2 \end{matrix}\right\} \quad (68)$$

These are the same results as in Eq. (66).

Constraints on $\hat{\theta}_{K1}$ *and* $\hat{\theta}_{K3}$: $\Rightarrow \boxed{\hat{\theta}_{K1} = \dot{\hat{\theta}}_{K1} = \hat{\theta}_{K3} = \dot{\hat{\theta}}_{K3} = 0} \quad \Rightarrow \boxed{S_{K1} = S_{K3} = 0} \quad \Rightarrow \boxed{C_{K1} = C_{K3} = 1}$

In this case, $y_i \left(i = 3K-2, 3K,\ 3(N+K)-2,\ 3(N+K)-1,\ 3(N+K)\right)$ are ***known variables*** and ***equal*** to ***zero***, so they can be eliminated from the equations of motion. The first two Lagrange multipliers are associated with angles $\hat{\theta}_{K1}$ and $\hat{\theta}_{K3}$, and the constraint moments. As above, the last three are the J frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body J components,

$$\left\{\begin{matrix} \lambda_1 \\ 0 \\ \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ (S_{K2})M_1^{K/(J)} - (S_{K1}C_{K2})M_2^{K/(J)} + (C_{K1}C_{K2})M_3^{K/(J)} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ (S_{K2})M_1^{K/(J)} + (C_{K2})M_3^{K/(J)} \end{matrix}\right\} \quad (69)$$

Solving gives,

$$\left\{\begin{matrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ M_3^{K/(J)} \end{matrix}\right\} = \left\{\begin{matrix} \lambda_1 \\ 0 \\ \left(\lambda_2 - (S_{K2})\lambda_1\right)/C_{K2} \end{matrix}\right\} \qquad (70)$$

Using body K components,

$$\left\{\begin{matrix} \lambda_1 \\ 0 \\ \lambda_2 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K2})M_1^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} \qquad (71)$$

Solving gives,

$$\left\{\begin{matrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} \left(\lambda_1 - (S_{K2})\lambda_2\right)/C_{K2} \\ 0 \\ \lambda_2 \end{matrix}\right\} \qquad (72)$$

The K frame components can also be found directly from the J frame components using the transformation matrix $\left[{}^J R_K\right]$ which represents 2 rotation.

Constraints on $\hat{\theta}_{K1}$ *and* $\hat{\theta}_{K2}$: $\Rightarrow \boxed{\hat{\theta}_{K1} = \dot{\hat{\theta}}_{K1} = \hat{\theta}_{K2} = \dot{\hat{\theta}}_{K2} = 0} \quad \Rightarrow \boxed{S_{K1} = S_{K2} = 0} \quad \Rightarrow \boxed{C_{K1} = C_{K2} = 1}$

In this case, $y_i \left(i = 3K-2, 3K-1,\ 3(N+K)-2,\ 3(N+K)-1,\ 3(N+K)\right)$ are ***known variables*** and ***equal*** to ***zero***, so they can be eliminated from the equations of motion. The first two Lagrange multipliers are associated with angles $\hat{\theta}_{K1}$ and $\hat{\theta}_{K2}$, and the constraint moments. As above, the last three are the J frame components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body J components,

$$\left\{\begin{matrix} \lambda_1 \\ \lambda_2 \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ (C_{K1})M_2^{K/(J)} + (S_{K1})M_3^{K/(J)} \\ (S_{K2})M_1^{K/(J)} - (S_{K1}C_{K2})M_2^{K/(J)} + (C_{K1}C_{K2})M_3^{K/(J)} \end{matrix}\right\} = \left\{\begin{matrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ M_3^{K/(J)} \end{matrix}\right\} \qquad (73)$$

Using body K components,

$$\left\{\begin{matrix} \lambda_1 \\ \lambda_2 \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K2}C_{K3})M_1^{(K)/J} - (C_{K2}S_{K3})M_2^{(K)/J} + (S_{K2})M_3^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} = \left\{\begin{matrix} (C_{K3})M_1^{(K)/J} - (S_{K3})M_2^{(K)/J} \\ (S_{K3})M_1^{(K)/J} + (C_{K3})M_2^{(K)/J} \\ M_3^{(K)/J} \end{matrix}\right\} \qquad (74)$$

Solving gives,

$$\boxed{\begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \end{Bmatrix} = \begin{bmatrix} C_{K3} & -S_{K3} \\ S_{K3} & C_{K3} \end{bmatrix}^{-1} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{bmatrix} C_{K3} & S_{K3} \\ -S_{K3} & C_{K3} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix}} \quad \boxed{M_3^{(K)/J} = 0} \tag{75}$$

The K frame components can also be found directly from the J frame components using the transformation matrix $\left[{}^J R_K \right]$ that represents a 2 rotation.

Prismatic Joints

Prismatic joints ***eliminate all rotational degrees*** of ***freedom*** and ***two*** of the three ***translational degrees*** of ***freedom*** between adjoining bodies. Hence, these joints allow only a single degree of translational freedom. For example, consider a joint that allows s'_{K1} to be free while the following constraints apply.

$$\boxed{\begin{Bmatrix} \hat{\theta}_{K1} \\ \hat{\theta}_{K2} \\ \hat{\theta}_{K3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}} \quad \text{and} \quad \boxed{\begin{Bmatrix} s'_{K2} \\ s'_{K3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

This represents a cylindrical joint along the $\underset{\sim}{n}_{J1}$ direction.

In this case, $y_i \; \left(i = 3K-2, 3K-1,\; 3K,\; 3(N+K)-1,\; 3(N+K) \right)$ are ***known variables*** and ***equal*** to ***zero***, so they can be eliminated from the equations of motion. The first three Lagrange multipliers are associated with angles $\hat{\theta}_{Ki} \;\; (i = 1,2,3)$ and the constraint moments. The last two multipliers are the $\underset{\sim}{n}_{J2}$ and $\underset{\sim}{n}_{J3}$ components of the constraint force. To find the components of the constraint moments, consider the right sides of equations $3K-2$, $3K-1$, and $3K$ associated with the constraints.

Using body J components,

$$\boxed{\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ (C_{K1}) M_2^{K/(J)} + (S_{K1}) M_3^{K/(J)} \\ (S_{K2}) M_1^{K/(J)} - (S_{K1} C_{K2}) M_2^{K/(J)} + (C_{K1} C_{K2}) M_3^{K/(J)} \end{Bmatrix} = \begin{Bmatrix} M_1^{K/(J)} \\ M_2^{K/(J)} \\ M_3^{K/(J)} \end{Bmatrix}}$$

Using body K components,

$$\boxed{\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} F^c_{y_{3K-2}} \\ F^c_{y_{3K-1}} \\ F^c_{y_{3K}} \end{Bmatrix} = \begin{Bmatrix} (C_{K2} C_{K3}) M_1^{(K)/J} - (C_{K2} S_{K3}) M_2^{(K)/J} + (S_{K2}) M_3^{(K)/J} \\ (S_{K3}) M_1^{(K)/J} + (C_{K3}) M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix} = \begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix}}$$

In this case, the J and K frames are ***aligned***, so the Lagrange multipliers are the components of the constraint moment in those frames.

Cylindrical Joints

Cylindrical joints ***eliminate two*** of the three ***rotational degrees*** of ***freedom*** and ***two*** of the three ***translational degrees*** of ***freedom*** between adjoining bodies. Hence, these joints allow two degrees of freedom. For example, consider a joint that allows $\hat{\theta}_{K1}$ and s'_{K1} to be free while the following constraints apply.

$$\boxed{\begin{Bmatrix} \hat{\theta}_{K2} \\ \hat{\theta}_{K3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}} \quad \text{and} \quad \boxed{\begin{Bmatrix} s'_{K2} \\ s'_{K3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

This joint allows rotation and translation between the adjoining bodies about and along the $\underset{\sim}{n}_{J1}$ direction.

In this case, $y_i \; \left(i = 3K-1,\; 3K,\; 3(N+K)-1,\; 3(N+K)\right)$ are ***known variables*** and ***equal*** to ***zero***, so they can be eliminated from the equations of motion. The first two Lagrange multipliers are associated with angles $\hat{\theta}_{Ki} \; \left(i = 2,3\right)$ and the constraint moments. The last two multipliers are the $\underset{\sim}{n}_{J2}$ and $\underset{\sim}{n}_{J3}$ components of the constraint force. The components of the constraint moments are as described above for a hinge joint about the $\underset{\sim}{n}_{J1}$ direction.

J frame components: $\boxed{M_1^{K/(J)} = 0}$ $\boxed{\begin{Bmatrix} M_2^{K/(J)} \\ M_3^{K/(J)} \end{Bmatrix} = \begin{bmatrix} C_{K1} & S_{K1} \\ -S_{K1} & C_{K1} \end{bmatrix}^{-1} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{bmatrix} C_{K1} & -S_{K1} \\ S_{K1} & C_{K1} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix}}$

K frame components: $\boxed{\begin{Bmatrix} M_1^{(K)/J} \\ M_2^{(K)/J} \\ M_3^{(K)/J} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \lambda_1 \\ \lambda_2 \end{Bmatrix}}$

Use of Additional Reference Frames

As written, these notes use a ***single set*** of ***reference directions*** for each body of the system. For convenience, however, it is often useful to introduce ***additional sets*** of ***reference directions*** for some or all the bodies. Consider, for example, a typical body *K*. It may be helpful to have ***one set*** of ***directions*** that make it easy to define the ***geometry*** of the ***connecting joint*** between *K* and its lower-body *J*, and a ***second set*** of directions that make it easy to define its ***inertia matrix***. These frames are related by a constant transformation matrix that can be easily included in the analysis.

Use of Euler Parameters

In previous notes, either orientation angles or Euler parameters were used exclusively in the development of the equations of motion. However, it may be more practical to use them both for some systems. When the singularities associated with orientation angles can be avoided, they can be used. When that doesn't seem possible for a given application, Euler parameters can be used. For example, Euler parameters could be used for all free and spherical joints.

When both orientation angles and Euler parameters are used, the state vector $\{x_1\}$ as described above will contain a combination of orientation angles and Euler parameters, and the state vector $\{y_1\}$ will contain a combination of orientation angle derivatives and angular velocity components. The kinematical Eqs. (5) will have to be modified to include the relationship between the Euler parameter derivatives and the angular velocity components.