An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume II – Kinetics: Summary of Contents

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Unit 1 – Inertia Matrices (Dyadics), Angular Momentum and Kinetic Energy

This unit defines ***moments*** and ***products of inertia*** for rigid bodies and shows how they are used to form ***inertia matrices*** (or ***dyadics***). Inertia matrices are then used to calculate ***principal moments*** of ***inertia*** and ***principal directions***. More generally, it shows how to ***transform*** the ***components*** of ***inertia dyadics*** from ***one set*** of reference axes to ***another***. Finally, it defines ***angular momentum vectors*** and the ***kinetic energy function*** for rigid bodies and shows ***how*** to use inertia matrices to compute them.

An ***Addendum*** is included to discuss the special case of ***nondistinct*** (equal) ***principal moments*** of ***inertia*** and their ***associated eigenvectors***. The principal moments of inertia and the principal directions of a square prism are presented as an example.

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35	6	8

Unit 2 – Newton/Euler Equations of Motion

This unit presents the ***Newton/Euler*** equations of motion for rigid bodies. These equations relate the ***kinematics*** of a body to the ***forces*** and ***torques*** acting upon it. The application of these equations focuses on ***individual bodies*** within a dynamic system using ***free body diagrams***. The analysis requires working knowledge of ***force systems***, ***kinematics*** of rigid bodies, and ***angular momentum***. The equations that result from the application of this method may be ***algebraic*** equations, ***differential*** equations, or a ***combination*** of both.

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Unit 3 – Degrees of Freedom, Partial Velocities, and Generalized Forces

This unit defines the concepts of ***degrees of freedom***, ***generalized coordinates***, ***partial velocities***, ***partial angular velocities*** and ***generalized forces***. These concepts form an introduction to methods of treating systems with multiple bodies as ***systems*** rather than one body at a time as with the Newton/Euler equations of motion.

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Unit 4 – Principle of Virtual Work and Lagrange's Equations

This unit discusses the ***Principle of Virtual Work*** for static systems and ***Lagrange's Equations*** for dynamic systems. The application of these methods relies heavily on the concepts of ***degrees of freedom***, ***generalized coordinates***, ***partial velocities***, ***partial angular velocities,*** and ***generalized forces*** discussed in Unit 3. As discussed in Unit 3, systems with multiple bodies will be analyzed as ***systems*** rather than one body at a time as with the Newton/Euler equations of motion.

Two addenda are provided to supplement the material presented in this unit. Addendum 1 shows the ***generalized forces*** associated with ***two equivalent force systems*** acting on a rigid body are ***equal***. Addendum 2 shows how to calculate the ***time derivative*** of a ***transformation matrix*** of a reference frame in terms of a ***skew-symmetric matrix*** related to the ***angular velocity*** of that frame.

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Unit 5 – d'Alembert's Principle and Kane's Equations

This unit discusses the use of ***d'Alembert's principle*** and ***Kane's equations*** to develop the equations of motion of rigid body dynamic systems. The application of these methods relies heavily on the concepts of ***degrees of freedom***, ***generalized coordinates***, ***partial velocities***, ***partial angular velocities,*** and ***generalized forces*** discussed in Unit 3. As discussed in Units 3 and 4, systems with multiple bodies will be analyzed as ***systems*** rather than one body at a time as with the Newton/Euler equations of motion. Examples include an aircraft with two engines and an upright, two-wheeled bicycle.

Page Count	Examples	Suggested Exercises
66	7	8

Unit 6 – Lagrange's Equations, d'Alembert's Principle, and Kane's Equations for Systems with Constraints

This unit discusses the application of Lagrange's equations, d'Alembert's principle and Kane's equations to rigid body dynamic systems with ***constraints***. In Units 4 and 5 of this volume, the applications of Lagrange's equations and d'Alembert's principle were based on a set of ***independent generalized coordinates***, and the applications of Kane's equations were based on a set of ***independent generalized speeds***. There are systems for which it is ***inconvenient*** or ***impossible*** to eliminate surplus generalized coordinates or ***simply inconvenient*** to eliminate generalized speeds from the analysis. For these systems, the application of each of these three methods can be supplemented with the use of ***Lagrange multipliers***.

The resulting equations of motion are a set of ***differential*** and ***algebraic*** equations. The Lagrange multipliers are ***algebraic unknowns*** related to the forces and torques required to maintain the constraints. An Addendum to this unit explores this connection.

In the analysis that follows, constraint equations are assumed to be ***equality constraints***, that is, some function of the generalized coordinates and/or generalized speeds is equal to zero. The case of inequality constraints is not considered.

Page Count	Examples	Suggested Exercises
30	4	7

Unit 7 – Introduction to Modeling Mechanical System Kinetics using MATLAB® Scripts, Simulink®, and SimMechanics®

This unit provides and introduction to ***modeling*** mechanical system kinetics using ***MATLAB scripts***, Simulink ***models***, and ***SimMechanics models***. For an introduction to these modeling techniques, see Unit 10 of Volume I for applications in mechanical systems kinematics. The examples presented in this unit assume the reader is familiar with the programming concepts presented in Volume I.

As presented in Volume I, MATLAB scripts are ***text-based programs*** written in the MATLAB programming language. Simulink models are ***block-diagram-based programs*** that run in the MATLAB environment. SimMechanics models are ***block-diagram-based, multibody dynamics programs*** that run in the MATLAB/Simulink environment. MATLAB scripts can be used ***alone*** or ***in conjunction with*** Simulink and SimMechanics models.

Three of the six models presented herein simulate the ***free motion*** of an ***upright bicycle***. The models are developed using the equations of motion developed in Unit 5 of this volume.

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Unit 8 – Basic Concepts of Linearization, Stability, Mode Shapes, and Natural Frequencies

The equations of motion of rigid body systems can be ***linear*** or ***nonlinear***. If the equations are ***nonlinear***, it may be possible, depending on the system, to ***linearize*** them about ***steady-state equilibrium*** conditions. Equilibrium conditions may involve constant positional coordinates, constant motion variables, or a combination of the two. This unit describes how to ***linearize*** equations of motion about ***equilibrium states*** and how to find ***equilibrium positions*** (if they exist). It also shows how to use the linear, approximate equations of motion to determine the ***stability*** of ***small motions*** about ***equilibrium states***. Finally, for ***stable equilibrium positions***, this unit shows how to calculate the ***undamped natural frequencies*** and ***mode shapes*** associated with those positions. Examples are given for one, two, and three degree-of-freedom systems. ***Extension*** to multi-degree-of-freedom systems is ***straightforward***.

MATLAB is used to calculate ***eigenvalues*** and ***eigenvectors*** associated with ***stability*** and ***modal analyses*** of systems based on their ***linearized equations*** of ***motion***. It is also used to ***solve nonlinear differential equations*** of ***motion*** as a means of ***verifying*** the ***conclusions*** of ***stability*** analyses based on ***linearized equations*** of ***motion***. A MATLAB/Simulink ***model*** of ***tumbling motion*** is presented in the ***Addendum***.

Page Count	Examples	Models	Suggested Exercises
42	9	1	6

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