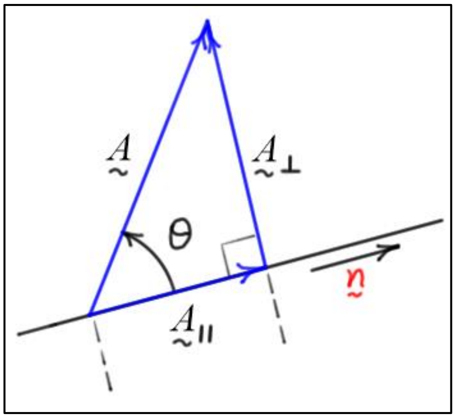
Elementary Statics
The Dot (or Scalar) Product of Two Vectors

Geometric Definition

The ***dot*** (or ***scalar***) product of two vectors is defined as follows:

$$\boxed{\underset{\sim}{A}\cdot\underset{\sim}{B}=|\underset{\sim}{A}|\,|\underset{\sim}{B}|\cos(\theta)}$$



Here, θ represents the ***angle*** between the two vectors. If one of the vectors is a ***unit vector***, the dot product is the other vector's ***projection*** in the direction of the unit vector.

$$\boxed{\underset{\sim}{A}\cdot\underset{\sim}{n}=|\underset{\sim}{A}|\,|\underset{\sim}{n}|\cos(\theta)=|\underset{\sim}{A}|\cos(\theta)}$$

The ***components*** of $\underset{\sim}{A}$ ***parallel*** and ***perpendicular*** to $\underset{\sim}{n}$ are

$$\boxed{\underset{\sim}{A}_{||}=\left(\underset{\sim}{A}\cdot\underset{\sim}{n}\right)\underset{\sim}{n}} \quad \text{and} \quad \boxed{\underset{\sim}{A}_{\perp}=\underset{\sim}{A}-\underset{\sim}{A}_{||}}$$

The ***dot product*** of two vectors is ***zero*** if they are ***perpendicular*** to each other.

Calculation

Given two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ expressed in terms of a ***mutually perpendicular set*** of ***unit vectors*** $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$, the ***dot product*** can be calculated as follows.

$$\boxed{\underset{\sim}{A}\cdot\underset{\sim}{B}=\left(a_x\,\underset{\sim}{i}+a_y\,\underset{\sim}{j}+a_z\,\underset{\sim}{k}\right)\cdot\left(b_x\,\underset{\sim}{i}+b_y\,\underset{\sim}{j}+b_z\,\underset{\sim}{k}\right)=a_xb_x+a_yb_y+a_zb_z}$$

Properties of the Dot Product

- Product is ***commutative***: $\boxed{\underset{\sim}{A}\cdot\underset{\sim}{B}=\underset{\sim}{B}\cdot\underset{\sim}{A}}$
- Product is ***distributive*** over addition: $\boxed{\underset{\sim}{A}\cdot\left(\underset{\sim}{B}+\underset{\sim}{C}\right)=\left(\underset{\sim}{A}\cdot\underset{\sim}{B}\right)+\left(\underset{\sim}{A}\cdot\underset{\sim}{C}\right)}$
- Multiplication by a ***scalar*** α: $\boxed{\alpha\left(\underset{\sim}{A}\cdot\underset{\sim}{B}\right)=\left(\alpha\underset{\sim}{A}\right)\cdot\underset{\sim}{B}=\underset{\sim}{A}\cdot\left(\alpha\underset{\sim}{B}\right)}$

Example #1:

Given: Two vectors, $\underset{\sim}{A} = 10\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + 8\underset{\sim}{k}$ and $\underset{\sim}{B} = 3\underset{\sim}{i} + 7\,\underset{\sim}{j} + 5\underset{\sim}{k}$

Find: The angle between the two vectors, θ.

Solution:

The angle can be calculated using the ***inverse cosine function*** as follows.

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|}\right) = \cos^{-1}\left(\frac{30+14+40}{\sqrt{10^2+2^2+8^2}\sqrt{3^2+7^2+5^2}}\right) \approx \cos^{-1}\left(\frac{84}{118.085}\right)$$
$$\approx 44.6548 \approx 44.7 \text{ (deg)}$$

Example #2:

Given: A vector, $\underset{\sim}{A} = 10\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + 8\underset{\sim}{k}$, and a ***unit vector*** $\underset{\sim}{n} = \left(\tfrac{2}{7}\right)\underset{\sim}{i} + \left(\tfrac{6}{7}\right)\underset{\sim}{j} + \left(\tfrac{3}{7}\right)\underset{\sim}{k}$.

Find: a) θ the angle between the two vectors, b) $\underset{\sim}{A}_{\parallel}$ the component of $\underset{\sim}{A}$ ***parallel*** to $\underset{\sim}{n}$, and c) $\underset{\sim}{A}_{\perp}$ the component of $\underset{\sim}{A}$ ***perpendicular*** to $\underset{\sim}{n}$.

Solution:

a) The angle can be calculated using the ***inverse cosine function*** as before.

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{A}\cdot\underset{\sim}{n}}{|\underset{\sim}{A}|}\right) = \cos^{-1}\left(\frac{(10\cdot\frac{2}{7})+(2\cdot\frac{6}{7})+(8\cdot\frac{3}{7})}{\sqrt{10^2+2^2+8^2}}\right) \approx \cos^{-1}\left(\frac{8}{12.9615}\right)$$
$$\approx 51.8871 \approx 51.9 \text{ (deg)}$$

b) $$\underset{\sim}{A}_{\parallel} = (\underset{\sim}{A}\cdot\underset{\sim}{n})\,\underset{\sim}{n} = 8\left[\left(\tfrac{2}{7}\right)\underset{\sim}{i} + \left(\tfrac{6}{7}\right)\underset{\sim}{j} + \left(\tfrac{3}{7}\right)\underset{\sim}{k}\right] \approx 2.29\,\underset{\sim}{i} + 6.86\,\underset{\sim}{j} + 3.43\underset{\sim}{k}$$

c) $$\underset{\sim}{A}_{\perp} = \underset{\sim}{A} - \underset{\sim}{A}_{\parallel} \approx \left(10\,\underset{\sim}{i} + 2\,\underset{\sim}{j} + 8\underset{\sim}{k}\right) - \left(2.29\underset{\sim}{i} + 6.86\,\underset{\sim}{j} + 3.43\underset{\sim}{k}\right) \approx 7.71\underset{\sim}{i} - 4.86\,\underset{\sim}{j} + 4.57\underset{\sim}{k}$$

Check:

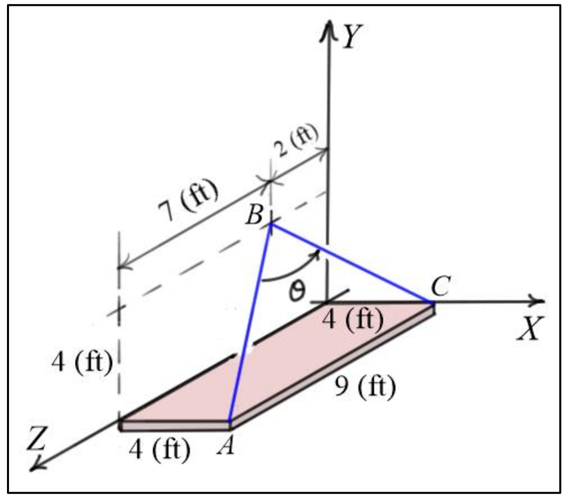
$$\underset{\sim}{A}_{\parallel}\cdot\underset{\sim}{A}_{\perp} \approx \left(2.29\underset{\sim}{i} + 6.86\,\underset{\sim}{j} + 3.43\underset{\sim}{k}\right)\cdot\left(7.71\underset{\sim}{i} - 4.86\,\underset{\sim}{j} + 4.57\underset{\sim}{k}\right)$$
$$\approx 17.6327 - 33.3061 + 15.6735 \approx 0$$

Example #3:

Given: Two support cables AB and CB are attached to the rectangular plate as shown.

Find: The angle θ between the two cables.

Solution:



The unit vectors pointing from A to B and C to B can be calculated using the dimensions shown in the figure as follows.

$$\underset{\sim}{u}_{AB} = \left(-4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}\right) \Big/ \sqrt{4^2+4^2+7^2} = -\tfrac{4}{9}\underset{\sim}{i} + \tfrac{4}{9}\underset{\sim}{j} - \tfrac{7}{9}\underset{\sim}{k}$$

$$\underset{\sim}{u}_{CB} = \left(-4\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}\right) \Big/ \sqrt{4^2+4^2+2^2} = -\tfrac{4}{6}\underset{\sim}{i} + \tfrac{4}{6}\underset{\sim}{j} + \tfrac{2}{6}\underset{\sim}{k} = -\tfrac{2}{3}\underset{\sim}{i} + \tfrac{2}{3}\underset{\sim}{j} + \tfrac{1}{3}\underset{\sim}{k}$$

Using the dot product, the angle θ can be calculated as follows.

$$\theta = \cos^{-1}\left(\underset{\sim}{u}_{AB} \cdot \underset{\sim}{u}_{CB}\right) = \cos^{-1}\left[\left(-\tfrac{4}{9}\right)\left(-\tfrac{2}{3}\right) + \left(\tfrac{4}{9}\right)\left(\tfrac{2}{3}\right) + \left(-\tfrac{7}{9}\right)\left(\tfrac{1}{3}\right)\right] = \cos^{-1}\left(\tfrac{16-7}{27}\right)$$

$$= \cos^{-1}\left(\tfrac{1}{3}\right)$$

$$\Rightarrow \boxed{\theta \approx 70.5 \text{ (deg)}}$$