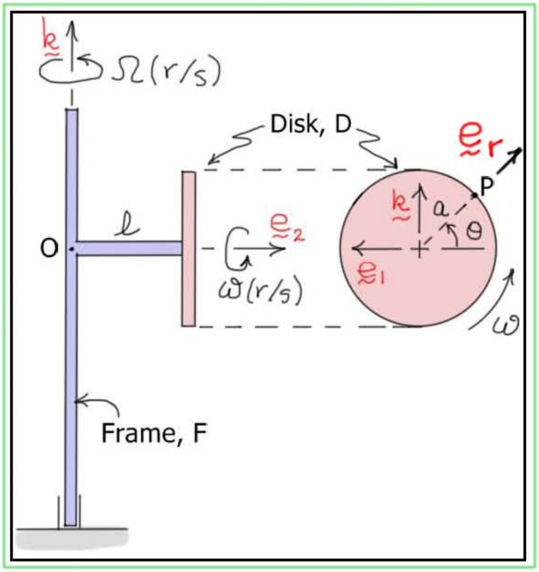
Example #2a – Intermediate Dynamics: Velocity



Reference frames:

$R:\ \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (fixed frame)

$F:\ \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k}$ (fixed in the rotating frame)

Find:

${}^{R}\underset{\sim}{v}_P$ … the ***velocity*** of point P in R using ***direct differentiation***

Solution:

To find the velocity of P, differentiate the position vector of P. Here, $\underset{\sim}{e}_r$ is a unit vector pointing from the center of the disk towards P.

$$ {}^{R}\underset{\sim}{v}_P = \frac{{}^{R}d}{dt}\left(\underset{\sim}{r}_{P/O}\right) = \frac{{}^{R}d}{dt}\left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right) = \ell\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_2\right) + a\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_r\right) \quad \text{(expressed in mixed frames)} $$

$$ = \ell\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{e}_2\right) + a\left[\left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right)\times\underset{\sim}{e}_r\right] = -\ell\,\Omega\,\underset{\sim}{e}_1 + a\left[\left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right)\times\underbrace{\left(-C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{k}\right)}_{\underset{\sim}{e}_r}\right] $$

$$ = -\ell\,\Omega\,\underset{\sim}{e}_1 + a\left[\omega\,C_\theta\,\underset{\sim}{k} + S_\theta\,\omega\,\underset{\sim}{e}_1 - \Omega C_\theta\,\underset{\sim}{e}_2\right] $$

So,

$$ {}^{R}\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k} $$

(expressed in frame F)

Aside:

$$ \frac{d}{dt}\left(-aC_\theta\right) = -a\left(-S_\theta\,\dot{\theta}\right) = a\,\omega\,S_\theta $$

$$ \frac{d}{dt}\left(aS_\theta\right) = a\left(C_\theta\,\dot{\theta}\right) = a\,\omega\,C_\theta $$

Or,

$$ {}^{R}\underset{\sim}{v}_P = \frac{{}^{R}d}{dt}\left(\underset{\sim}{r}_{P/O}\right) = \frac{{}^{R}d}{dt}\left(-aC_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + aS_\theta\,\underset{\sim}{k}\right) \quad \text{(expressed in frame } F\text{)} $$

$$ = \left(aS_\theta\,\omega\right)\underset{\sim}{e}_1 - aC_\theta\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_1\right) + \ell\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_2\right) + \left(aC_\theta\omega\right)\underset{\sim}{k} + \left(aS_\theta\right)\underbrace{\frac{{}^{R}d}{dt}\left(\underset{\sim}{k}\right)}_{\text{zero}} $$

$$ = \left(aS_\theta\,\omega\right)\underset{\sim}{e}_1 - aC_\theta\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{e}_1\right) + \ell\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{e}_2\right) + \left(aC_\theta\omega\right)\underset{\sim}{k} $$

$$ = \left(aS_\theta\,\omega\right)\underset{\sim}{e}_1 - aC_\theta\,\Omega\,\underset{\sim}{e}_2 - \ell\,\Omega\,\underset{\sim}{e}_1 + \left(aC_\theta\omega\right)\underset{\sim}{k} $$

So,

$$ {}^{R}\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k} $$