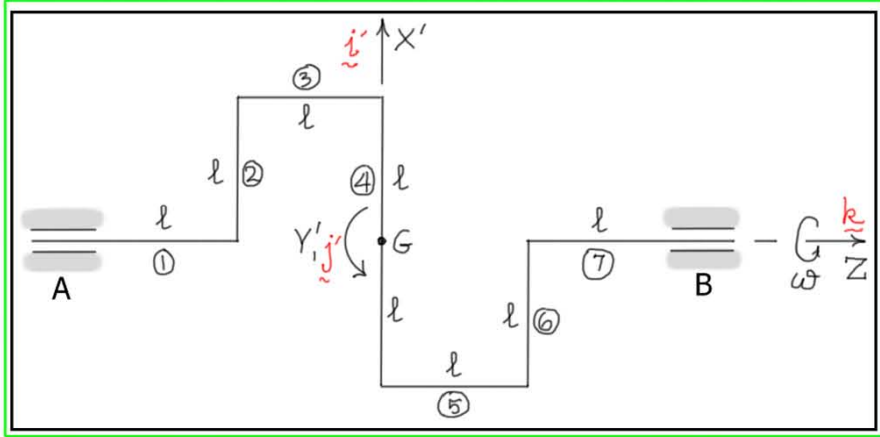


Example #11 – Intermediate Dynamics: Kinetic Energy

The figure shows a **simple crank shaft** consisting of **seven segments**, each considered to be a **slender bar**. Each segment of **length** ℓ has **mass** m . There are six segments of length ℓ and one segment of length 2ℓ (segment 4). The mass center of the system is G and is located on the axis of rotation.



Reference frames:

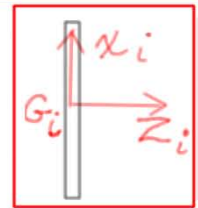
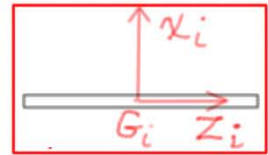
$R: \underline{i}, \underline{j}, \underline{k}$ (fixed frame)

$S: \underline{i}', \underline{j}', \underline{k}$ (rotates with the shaft)

Find:

\underline{H}_G ... angular momentum of the system about its mass center, G

K ... kinetic energy of the system



Solution: (components in rotating shaft frame)

$$\begin{Bmatrix} \underline{H}_G \cdot \underline{i}' \\ \underline{H}_G \cdot \underline{j}' \\ \underline{H}_G \cdot \underline{k} \end{Bmatrix} = \begin{bmatrix} I_{X'X'}^G & -I_{X'Y'}^G & -I_{X'Z}^G \\ -I_{Y'X'}^G & I_{Y'Y'}^G & -I_{Y'Z}^G \\ -I_{Z'X'}^G & -I_{Z'Y'}^G & I_{ZZ}^G \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I_{X'Z}^G \omega \\ -I_{Y'Z}^G \omega \\ I_{ZZ}^G \omega \end{Bmatrix}$$

$$I_{ii}^A = I_{ii}^G + m d_i^2 \quad (i = x, y, \text{ or } z)$$

$$I_{ij}^A = I_{ij}^G + m c_i c_j \quad (i \text{ and } j = x, y, \text{ or } z)$$

Using the parallel axes theorems for moments and products of inertia: (parallel axes theorems)

$$I_{ZZ}^G = \sum_{i=1}^7 (I_{ZZ}^G)_i = 0 + \frac{1}{3}m\ell^2 + m\ell^2 + \frac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \frac{1}{3}m\ell^2 + 0 = \frac{10}{3}m\ell^2$$

$$I_{X'Z}^G = \sum_{i=1}^7 (I_{X'Z}^G)_i = 0 + m(\frac{\ell}{2})(-\ell) + m(\ell)(-\frac{\ell}{2}) + 0 + m(-\ell)(\frac{\ell}{2}) + m(-\frac{\ell}{2})(\ell) + 0 = -2m\ell^2$$

$$I_{Y'Z}^G \equiv 0 \quad (\text{since the } X'Z \text{ plane is a plane of symmetry, so products associated with } Y' \text{ are zero})$$

So,

$$\underline{H}_G = 2m\ell^2\omega \underline{i}' + (\frac{10}{3})m\ell^2\omega \underline{k}$$

The **kinetic energy** of the crank shaft is found from the **velocity** and **angular momentum** vectors to be

$$K = \underbrace{\frac{1}{2}m(\underline{v}_G^R)^2}_{\text{zero}} + \frac{1}{2} {}^R\omega_B \cdot \underline{H}_G = \frac{1}{2} {}^R\omega_B \cdot \underline{H}_G = \frac{1}{2}(\omega \underline{k}) \cdot \underline{H}_G = \frac{10}{6}m\ell^2\omega^2$$