

## Example 12 – Intermediate Dynamics: Angular Momentum and Kinetic Energy

Reference frames: ( $R$  is the fixed frame)

$S$ :  $\underline{i}', \underline{j}', \underline{k}$  (rotates with the shaft; aligned with the shaft)

$D$ :  $\underline{i}', \underline{e}_2, \underline{e}_3$  (rotates with the shaft; aligned with the disk)

Find:

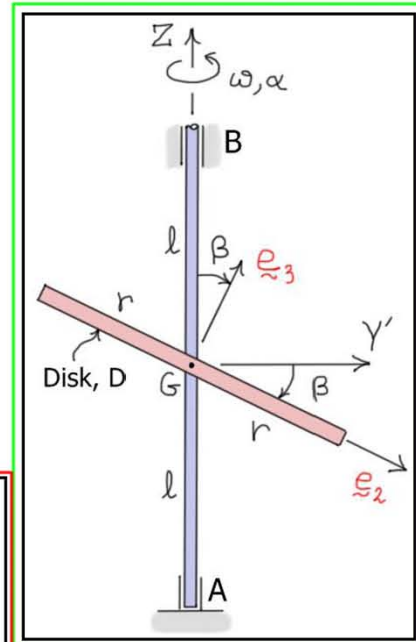
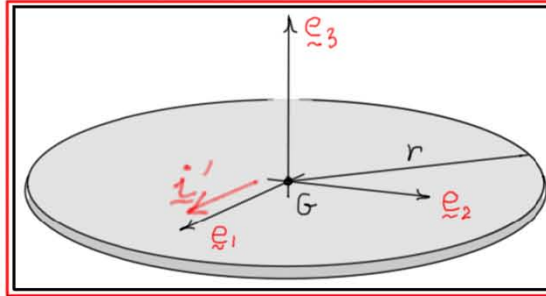
$\underline{H}_G$  ... **angular momentum** of the disk about its mass center,  $G$

$K$  ... **kinetic energy** of the disk

Solution:

$${}^R\omega_D = \omega \underline{k} = \omega (-S_\beta \underline{e}_2 + C_\beta \underline{e}_3)$$

$$[I_G]_D = mr^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$



$$\begin{Bmatrix} \underline{H}_G \cdot \underline{i}' \\ \underline{H}_G \cdot \underline{e}_2 \\ \underline{H}_G \cdot \underline{e}_3 \end{Bmatrix} = mr^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ -\omega S_\beta \\ \omega C_\beta \end{Bmatrix} = mr^2 \begin{Bmatrix} 0 \\ -\frac{1}{4}\omega S_\beta \\ \frac{1}{2}\omega C_\beta \end{Bmatrix} \Rightarrow \underline{H}_G = \frac{1}{4}mr^2\omega (-S_\beta \underline{e}_2 + 2C_\beta \underline{e}_3)$$

Expressing  $\underline{H}_G$  in the reference frame  $S$ :

$$\underline{H}_G = \frac{1}{4}mr^2\omega (-S_\beta \underline{e}_2 + 2C_\beta \underline{e}_3) = \frac{1}{4}mr^2\omega [-S_\beta (C_\beta \underline{j}' - S_\beta \underline{k}) + 2C_\beta (S_\beta \underline{j}' + C_\beta \underline{k})]$$

Or,

$$\underline{H}_G = \frac{1}{4}mr^2\omega [(S_\beta C_\beta) \underline{j}' + (2C_\beta^2 + S_\beta^2) \underline{k}] = \frac{1}{4}mr^2\omega [(S_\beta C_\beta) \underline{j}' + (C_\beta^2 + 1) \underline{k}]$$

We also know that  $\underline{H}_G = (-I_{X'Z}\omega) \underline{i}' + (-I_{Y'Z}\omega) \underline{j}' + (I_{ZZ}\omega) \underline{k}$ .

Comparing these two results, we find:  $I_{X'Z} = 0$ ,  $I_{Y'Z} = -\frac{1}{4}mr^2 S_\beta C_\beta$ ,  $I_{ZZ} = \frac{1}{4}mr^2 (C_\beta^2 + 1)$

The **kinetic energy** of the disk is found from the **velocity** and **angular momentum** vectors:

$$K = \underbrace{\frac{1}{2}m({}^R\underline{v}_G)^2}_{\text{zero}} + \frac{1}{2}{}^R\omega_D \cdot \underline{H}_G = \frac{1}{2}{}^R\omega_D \cdot \underline{H}_G = \frac{1}{2}(\omega \underline{k}) \cdot \underline{H}_G = \frac{1}{8}mr^2 (C_\beta^2 + 1)\omega^2$$