

## Example #13 – Intermediate Dynamics: Angular Momentum and Kinetic Energy

Reference frames: ( $R$  is the fixed frame)

$$F: (\underline{n}_1, \underline{n}_2, \underline{k}) \dots \text{(rotates with frame } F)$$

$$B: (\underline{e}_1, \underline{n}_2, \underline{e}_3) \dots \text{(rotates with the bar } B)$$

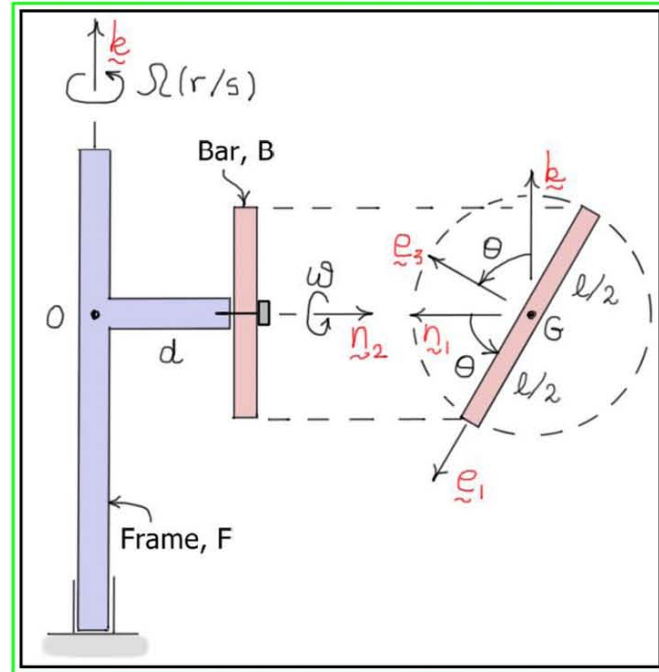
Find:

$$\underline{H}_G \dots \text{angular momentum of } B \text{ about its mass-center, } G$$

$$K \dots \text{kinetic energy of } B$$

Solution:

The **angular velocity vector** and the **inertia matrix** about **body-fixed axes** are:



$$[I_G]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m l^2 & 0 \\ 0 & 0 & \frac{1}{12} m l^2 \end{bmatrix}$$

$$\text{and } {}^R \underline{\omega}_B = \omega \underline{n}_2 + \Omega \underline{k} = \omega \underline{n}_2 + \Omega (-S_\theta \underline{e}_1 + C_\theta \underline{e}_3)$$

$$\Rightarrow {}^R \underline{\omega}_B = (-\Omega S_\theta) \underline{e}_1 + \omega \underline{n}_2 + (\Omega C_\theta) \underline{e}_3$$

The **body-fixed components** of the **angular momentum** vector are

$$\begin{Bmatrix} \underline{H}_G \cdot \underline{e}_1 \\ \underline{H}_G \cdot \underline{n}_2 \\ \underline{H}_G \cdot \underline{e}_3 \end{Bmatrix} = \frac{m l^2}{12} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m l^2}{12} \begin{Bmatrix} 0 \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m l^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3)$$

The **angular momentum** can also be expressed using unit vectors fixed in  $F$ :

$$\underline{H}_G = \frac{m l^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3) = \frac{m l^2}{12} [\omega \underline{n}_2 + \Omega C_\theta (S_\theta \underline{n}_1 + C_\theta \underline{k})] = \frac{m l^2}{12} [\Omega C_\theta S_\theta \underline{n}_1 + \omega \underline{n}_2 + \Omega C_\theta^2 \underline{k}]$$

The kinetic energy of the bar is calculated using the angular momentum vector as follows.

$$\begin{aligned} K &= K_{\text{translation}} + K_{\text{rotation}} = \frac{1}{2} m ({}^R \underline{v}_G)^2 + \frac{1}{2} {}^R \underline{\omega}_B \cdot \underline{H}_G \\ &= \frac{1}{2} m d^2 \Omega^2 + \frac{1}{2} (-\Omega S_\theta \underline{e}_1 + \omega \underline{n}_2 + \Omega C_\theta \underline{e}_3) \cdot \frac{m l^2}{12} (\omega \underline{n}_2 + \Omega C_\theta \underline{e}_3) \end{aligned}$$

$$\Rightarrow K = \frac{1}{2} m d^2 \Omega^2 + \frac{m l^2}{24} (\omega^2 + \Omega^2 C_\theta^2)$$