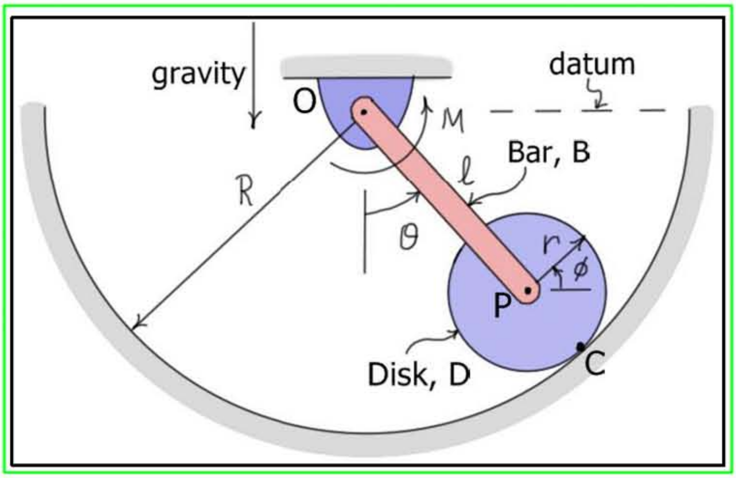
Example #20 – Intermediate Dynamics: Lagrange's Equations (1 DOF system)



Given:

$m_B = m_D = m$

D ***rolls without slipping*** on the circular surface

Active Forces/Torques:

Torque M and weight forces

Find:

Differential equation of motion of the system

Solution: (using θ as the generalized coordinate)

Using θ as the single ***generalized coordinate***, the ***equation of motion*** of the system can be found using Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F_\theta \quad \text{with} \quad L = K - V = K_B + K_D - V_B - V_D \quad \text{(Lagrangian)}$$

$$K_D = \tfrac{1}{2}\underset{\sim}{\omega}_D \cdot \underset{\sim}{H}_C = \tfrac{1}{2} I_C \dot{\phi}^2 = \tfrac{1}{2}(\tfrac{1}{2}mr^2 + mr^2)\dot{\phi}^2 = \boxed{\tfrac{3}{4}mr^2\dot{\phi}^2} \quad \text{(fixed axis rotation)}$$

$$K_B = \tfrac{1}{2}\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_O = \tfrac{1}{2} I_O \dot{\theta}^2 = \tfrac{1}{2}(\tfrac{1}{3}m\ell^2)\dot{\theta}^2 = \boxed{\tfrac{1}{6}m\ell^2\dot{\theta}^2} \quad \text{(fixed axis rotation)}$$

$$V = V_D + V_B = -mg\ell C_\theta - \tfrac{1}{2}mg\ell C_\theta = \boxed{-\tfrac{3}{2}mg\ell C_\theta} \quad \text{(datum as shown in diagram)}$$

Before using Lagrange's equation, angle ϕ must be eliminated from the equations. Using the concept of instantaneous centers of zero velocity, note that $\boxed{v_P = \ell\dot{\theta} = -r\dot{\phi}}$, or $\boxed{\dot{\phi} = -(\ell / r)\dot{\theta}}$.

$$L = \tfrac{3}{4}m r^2 (\ell / r)^2 \dot{\theta}^2 + \tfrac{1}{6}m\ell^2\dot{\theta}^2 + \tfrac{3}{2}mg\ell C_\theta = \tfrac{11}{12}m\ell^2\dot{\theta}^2 + \tfrac{3}{2}mg\ell C_\theta$$

Generalized active force F_θ and the derivatives of the Lagrangian:

$$F_\theta = M\underset{\sim}{k} \cdot \frac{\partial}{\partial \dot{\theta}}(\underset{\sim}{\omega}_B) = M\underset{\sim}{k} \cdot \underset{\sim}{k} = M(t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \tfrac{11}{6}m\ell^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \tfrac{11}{6}m\ell^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -\tfrac{3}{2}mg\ell S_\theta$$

Substituting into Lagrange's equation gives the equation of motion

$$\boxed{\tfrac{11}{6}m\ell^2\ddot{\theta} + \tfrac{3}{2}mg\ell S_\theta = M(t)} \quad \text{(non-linear, second-order, ordinary, differential equation)}$$