

Example #21 – Intermediate Dynamics: Lagrange's Equations (2 DOF system)

Given:

driving torques M_1 and M_2

Active Forces/Torques:

M_1 , M_2 , and weight forces

Find:

differential equations of motion of the system

Solution: (using θ_1 and θ_2 as the generalized coordinates)

Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = F_{\theta_1} \quad \text{and} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = F_{\theta_2}$$

Kinematics:

$$\underline{v}_{G_1} = \underline{v}_O + \underline{v}_{G_1/O} = \frac{1}{2} \ell \dot{\theta}_1 \underline{e}_{\theta_1} \quad \Rightarrow \quad \underline{v}_{G_1}^2 = \underline{v}_{G_1} \cdot \underline{v}_{G_1} = \frac{1}{4} \ell^2 \dot{\theta}_1^2$$

$$\underline{v}_{G_2} = \underline{v}_A + \underline{v}_{G_2/A} = \underline{v}_O + \underline{v}_{A/O} + \underline{v}_{G_2/A} = \ell \dot{\theta}_1 \underline{e}_{\theta_1} + \frac{1}{2} \ell \dot{\theta}_2 \underline{e}_{\theta_2}$$

$$\Rightarrow \underline{v}_{G_2}^2 = \underline{v}_{G_2} \cdot \underline{v}_{G_2} = \ell^2 \dot{\theta}_1^2 + \frac{1}{4} \ell^2 \dot{\theta}_2^2 + 2 \left(\frac{1}{2} \ell^2 \dot{\theta}_1 \dot{\theta}_2 \right) (\underline{e}_{\theta_1} \cdot \underline{e}_{\theta_2}) = \ell^2 \dot{\theta}_1^2 + \frac{1}{4} \ell^2 \dot{\theta}_2^2 + \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1}$$

Lagrangian: $L = K - V = K_1 + K_2 - V_1 - V_2$

$$K_1 = \frac{1}{2} I_O \dot{\theta}_1^2 = \frac{1}{2} \left(\frac{1}{3} m \ell^2 \right) \dot{\theta}_1^2 = \frac{1}{6} m \ell^2 \dot{\theta}_1^2 \quad (\text{fixed axis rotation})$$

$$K_2 = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} I_{G_2} \dot{\theta}_2^2 = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{8} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} + \frac{1}{24} m \ell^2 \dot{\theta}_2^2 \quad (\text{general plane motion})$$

$$= \left[\frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{6} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} \right]$$

$$V = V_1 + V_2 = -\frac{1}{2} m g \ell C_1 - m g \left(\ell C_1 + \frac{1}{2} \ell C_2 \right) = -\frac{3}{2} m g \ell C_1 - \frac{1}{2} m g \ell C_2 \quad (\text{datum at O})$$

$$L = \frac{2}{3} m \ell^2 \dot{\theta}_1^2 + \frac{1}{6} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} + \frac{3}{2} m g \ell C_1 + \frac{1}{2} m g \ell C_2$$

Generalized Forces and Derivatives of Lagrangian:

$$F_{\theta_1} = \left(M_1 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(-M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_2}{\partial \dot{\theta}_1} \right) = M_1 - M_2$$

$$F_{\theta_2} = \left(M_1 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(-M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_2}{\partial \dot{\theta}_2} \right) = M_2$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{4}{3} m \ell^2 \dot{\theta}_1 + \frac{1}{2} m \ell^2 \dot{\theta}_2 C_{2-1} \quad \Rightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = \frac{4}{3} m \ell^2 \ddot{\theta}_1 + \frac{1}{2} m \ell^2 C_{2-1} \ddot{\theta}_2 - \frac{1}{2} m \ell^2 \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) S_{2-1}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} m \ell^2 C_{2-1} \dot{\theta}_1 + \frac{1}{3} m \ell^2 \dot{\theta}_2 \quad \Rightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = \frac{1}{2} m \ell^2 C_{2-1} \ddot{\theta}_1 + \frac{1}{3} m \ell^2 \ddot{\theta}_2 - \frac{1}{2} m \ell^2 \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1) S_{2-1}$$

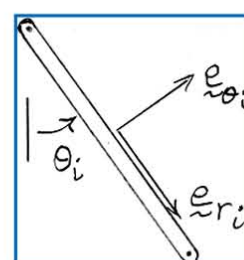
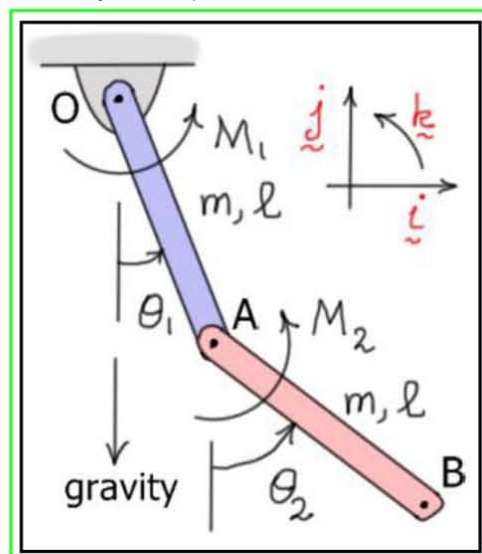
$$\frac{\partial L}{\partial \theta_1} = \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \frac{3}{2} m g \ell S_1$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \frac{1}{2} m g \ell S_2$$

Substituting into Lagrange's equations gives a **coupled** set of **nonlinear, second-order, ordinary differential equations of motion**

$$\left(\frac{4}{3} m \ell^2 \right) \ddot{\theta}_1 + \left(\frac{1}{2} m \ell^2 C_{2-1} \right) \ddot{\theta}_2 - \left(\frac{1}{2} m \ell^2 S_{2-1} \right) \dot{\theta}_2^2 + \frac{3}{2} m g \ell S_1 = M_1(t) - M_2(t)$$

$$\left(\frac{1}{2} m \ell^2 C_{2-1} \right) \ddot{\theta}_1 + \left(\frac{1}{3} m \ell^2 \right) \ddot{\theta}_2 + \left(\frac{1}{2} m \ell^2 S_{2-1} \right) \dot{\theta}_1^2 + \frac{1}{2} m g \ell S_2 = M_2(t)$$



Typical Link