

## Example #22 – Intermediate Dynamics: Lagrange’s Equations (2 DOF system)

Given:

$$M_1 = -k_1\theta_1 - c_1\dot{\theta}_1$$

$$M_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

Active Forces/Torques:

$$\underline{M}_1, \underline{M}_2, \text{ weight forces}$$

Find:

- **differential equations of motion** of the system

Solution: (using  $\theta_1$  and  $\theta_2$  as generalized coordinates)

Previous Results:

$$\left(\frac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 = M_1(t) - M_2(t)$$

$$\left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 = M_2(t)$$

Substituting for the torques  $M_1(t)$  and  $M_2(t)$  gives:

$$\left(\frac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 = -k_1\theta_1 - c_1\dot{\theta}_1 + k_2(\theta_2 - \theta_1) + c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

$$\left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

or

$$\left(\frac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0$$

$$\left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 + c_2(\dot{\theta}_2 - \dot{\theta}_1) + k_2(\theta_2 - \theta_1) = 0$$

As before, these equations represent a *coupled* set of nonlinear, second-order, ordinary differential equations of motion.

