Example #23 – Intermediate Dynamics: (Lagrange's Equations – 3D, 2 DOF system)

Reference frames: (R is the fixed frame)

$F:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$ … (rotates with frame F)

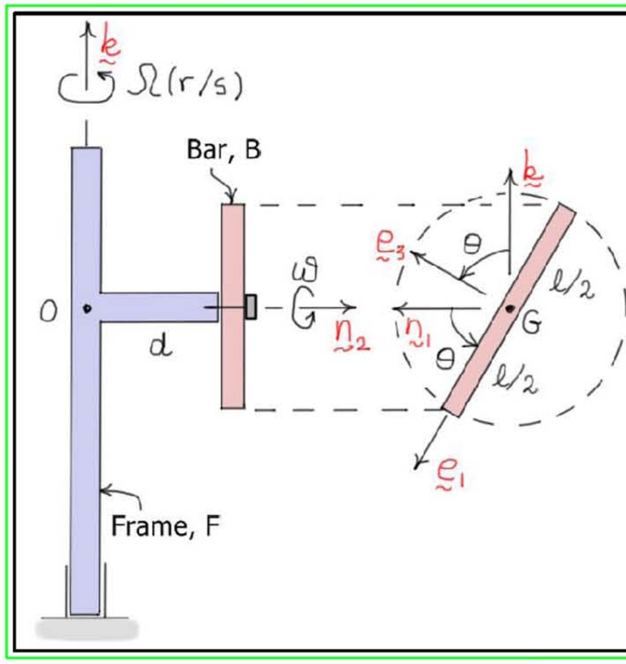
$B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$ … (rotates with the bar B)

Given:

$M_\phi(t)$ … applied to F by the ground

$M_\theta(t)$ … applied to B by F

B has mass m; F is light



Find:

- ***differential equations of motion*** of the system

Solution: (using ϕ and θ as ***generalized coordinates***)

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right)-\frac{\partial L}{\partial \phi}=F_\phi \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)-\frac{\partial L}{\partial \theta}=F_\theta$$

Previous Results:

$${}^R\underset{\sim}{\omega}_F=\dot\phi\,\underset{\sim}{k} \qquad {}^R\underset{\sim}{\omega}_B=\dot\theta\,\underset{\sim}{n}_2+\dot\phi\,\underset{\sim}{k}$$

$$K=\tfrac{1}{2}md^2\dot\phi^2+\frac{m\ell^2}{24}\left(\dot\theta^2+\dot\phi^2C_\theta^2\right)$$

Here,

$$L=K-V=K=\tfrac{1}{2}md^2\dot\phi^2+\frac{m\ell^2}{24}\left(\dot\theta^2+\dot\phi^2C_\theta^2\right)$$ (no potential energy term)

Generalized Forces: (associated with the driving torques)

$$F_\phi=\left(M_\theta\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot\phi}\right)+\left(-M_\theta\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot\phi}\right)+\left(M_\phi\underset{\sim}{k}\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot\phi}\right)=\boxed{M_\phi}$$

$$F_\theta=\left(M_\theta\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_B}{\partial\dot\theta}\right)+\left(-M_\theta\underset{\sim}{n}_2\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot\theta}\right)+\left(M_\phi\underset{\sim}{k}\cdot\frac{\partial\,{}^R\underset{\sim}{\omega}_F}{\partial\dot\theta}\right)=\boxed{M_\theta}$$

Derivatives of Lagrangian:

$$\frac{\partial L}{\partial\dot\phi}=md^2\dot\phi+\tfrac{1}{12}m\ell^2\dot\phi C_\theta^2 \quad\Rightarrow\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)=\left(md^2+\tfrac{1}{12}m\ell^2C_\theta^2\right)\ddot\phi-\tfrac{1}{6}m\ell^2\dot\theta\dot\phi S_\theta C_\theta$$

$$\frac{\partial L}{\partial\dot\theta}=\tfrac{1}{12}m\ell^2\dot\theta \quad\Rightarrow\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)=\tfrac{1}{12}m\ell^2\ddot\theta$$

$$\frac{d}{dt}\left(C_\theta^2\right)=2C_\theta(-S_\theta)\dot\theta=-2S_\theta C_\theta\dot\theta$$

$$\frac{\partial L}{\partial\phi}=0 \qquad \frac{\partial L}{\partial\theta}=-\tfrac{1}{12}m\ell^2\dot\phi^2S_\theta C_\theta$$

Equations of Motion:

Substituting into Lagrange's equations gives a ***coupled*** set of ***nonlinear, second-order, ordinary differential equations of motion***

$$\left(md^2+\tfrac{1}{12}m\ell^2C_\theta^2\right)\ddot\phi-\left(\tfrac{1}{6}m\ell^2S_\theta C_\theta\right)\dot\theta\dot\phi=M_\phi(t)$$

$$\left(\tfrac{1}{12}m\ell^2\right)\ddot\theta+\left(\tfrac{1}{12}m\ell^2S_\theta C_\theta\right)\dot\phi^2=M_\theta(t)$$