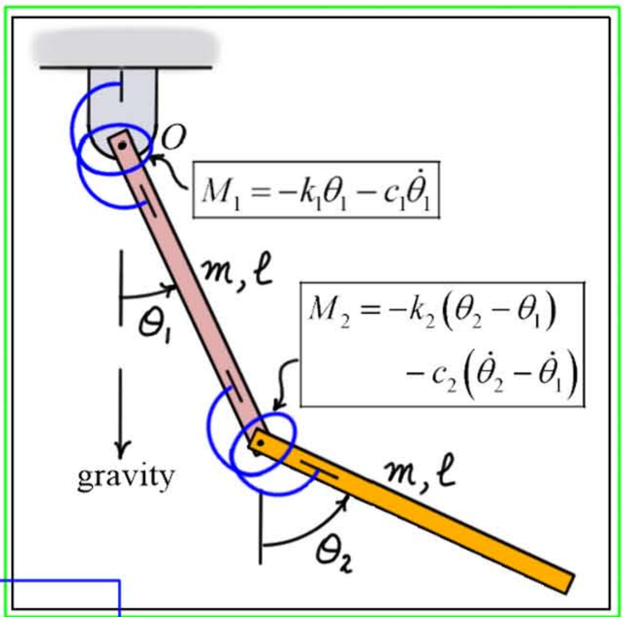
Example #25 – Intermediate Dynamics: (Linearization of Equations of Motion)

Given:

Equations of motion of the double pendulum with adjoining springs and dampers



$$\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1+\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2-\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2+\tfrac{3}{2}mg\ell S_1 +(c_1+c_2)\dot{\theta}_1-c_2\dot{\theta}_2+(k_1+k_2)\theta_1-k_2\theta_2=0$$

$$\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1+\left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2+\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 +\tfrac{1}{2}mg\ell S_2+c_2\left(\dot{\theta}_2-\dot{\theta}_1\right)+k_2\left(\theta_2-\theta_1\right)=0$$

Find:

- ***equilibrium position*** of the system
- ***approximate linear*** equation of motion about that position.

Solution:

The equilibrium positions of the system are found from the differential equations by setting

$$\dot{\theta}_1=\dot{\theta}_2=\ddot{\theta}_1=\ddot{\theta}_2=0$$

$$\Rightarrow \tfrac{3}{2}mg\ell S_1+(k_1+k_2)\theta_1-k_2\theta_2=0 \text{ and } \tfrac{1}{2}mg\ell S_2+k_2(\theta_2-\theta_1)=0 \quad\Rightarrow\quad \theta_1=\theta_2=0$$

Linearization of first equation about $\theta_1=\theta_2=0$: (spring and damper terms are linear)

1st term: $\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1$ … linear

2nd term: $\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 \Rightarrow f(\theta_1,\theta_2,\ddot{\theta}_2)=\ddot{\theta}_2\cos(\theta_2-\theta_1) \Rightarrow \Delta f\approx m_1\Delta\theta_1+m_2\Delta\theta_2+m_3\Delta\ddot{\theta}_2$

$$m_1=\left(\partial f/\partial\theta_1\right)_{eq}=\left[\ddot{\theta}_2\sin(\theta_2-\theta_1)\right]_{eq}=0 \qquad m_2=\left(\partial f/\partial\theta_2\right)_{eq}=\left[-\ddot{\theta}_2\sin(\theta_2-\theta_1)\right]_{eq}=0$$

$$m_3=\left(\partial f/\partial\ddot{\theta}_2\right)_{eq}=\left[\cos(\theta_2-\theta_1)\right]_{eq}=1 \quad\Rightarrow\quad \Delta f\approx\Delta\ddot{\theta}_2$$

3rd term: $\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 \Rightarrow f(\theta_1,\theta_2,\dot{\theta}_2)=\dot{\theta}_2^2\sin(\theta_2-\theta_1) \Rightarrow \Delta f\approx m_1\Delta\theta_1+m_2\Delta\theta_2+m_3\Delta\dot{\theta}_2$

$$m_1=\left(\partial f/\partial\theta_1\right)_{eq}=\left[-\dot{\theta}_2^2\cos(\theta_2-\theta_1)\right]_{eq}=0 \qquad m_2=\left(\partial f/\partial\theta_2\right)_{eq}=\left[\dot{\theta}_2^2\cos(\theta_2-\theta_1)\right]_{eq}=0$$

$$m_3=\left(\partial f/\partial\dot{\theta}_2\right)_{eq}=\left[2\dot{\theta}_2\sin(\theta_2-\theta_1)\right]_{eq}=0 \quad\Rightarrow\quad \Delta f\approx 0$$

4th term: $\tfrac{3}{2}mg\ell S_1 \Rightarrow f(\theta_1)=\sin(\theta_1) \Rightarrow \Delta f\approx\left(\partial f/\partial\theta_1\right)_{eq}\Delta\theta_1=\left(\cos(\theta_1)\right)_{eq}\Delta\theta_1=\Delta\theta_1$

Carrying out a similar process for the second equation yields similar results, so the two ***approximate linear differential equations of motion*** are

$$\left(\tfrac{4}{3}m\ell^2\right)\Delta\ddot{\theta}_1+\left(\tfrac{1}{2}m\ell^2\right)\Delta\ddot{\theta}_2+(c_1+c_2)\Delta\dot{\theta}_1-c_2\Delta\dot{\theta}_2+\left(k_1+k_2+\tfrac{3}{2}mg\ell\right)\Delta\theta_1-k_2\Delta\theta_2=0$$

$$\left(\tfrac{1}{2}m\ell^2\right)\Delta\ddot{\theta}_1+\left(\tfrac{1}{3}m\ell^2\right)\Delta\ddot{\theta}_2-c_2\Delta\dot{\theta}_1+c_2\Delta\dot{\theta}_2-k_2\Delta\theta_1+\left(k_2+\tfrac{1}{2}mg\ell\right)\Delta\theta_2=0$$

Writing these two equations in ***matrix form***:

$$[M]\begin{Bmatrix}\Delta\ddot{\theta}_1\\ \Delta\ddot{\theta}_2\end{Bmatrix}+[C]\begin{Bmatrix}\Delta\dot{\theta}_1\\ \Delta\dot{\theta}_2\end{Bmatrix}+[K]\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Here,

$$[M]=\begin{bmatrix}\frac{4}{3}m\ell^2 & \frac{1}{2}m\ell^2\\ \frac{1}{2}m\ell^2 & \frac{1}{3}m\ell^2\end{bmatrix}$$

mass matrix

$$[C]=\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}$$

damping matrix

$$[K]=\begin{bmatrix}k_1+k_2+\frac{3}{2}mg\ell & -k_2\\ -k_2 & k_2+\frac{1}{2}mg\ell\end{bmatrix}$$

stiffness matrix