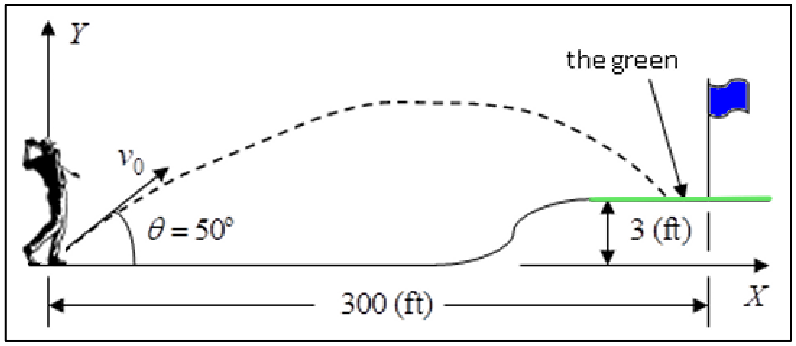
Elementary Engineering Mathematics
Application of Quadratic Equations in Elementary Dynamics

Example #1

Given: A ***golfer*** hits a ball with an ***initial velocity*** of $v_0 = 96\text{ (ft/s)}$ at an ***angle*** of $\theta = 50\text{ (deg)}$. If we neglect air resistance, the following equations describe the x and y positions of the ball (measured in feet) as a functions of time (t, sec). That is, they describe the ***path*** of the ball.



$$\boxed{x(t) = 61.71t} \qquad \boxed{y(t) = 73.54t - 16.1t^2} \tag{1}$$

Find: a) the times when $y = 50\text{ (ft)}$, b) how long it takes for the ball to land on the green, c) the x-coordinate of the ball when it lands on the green, d) the maximum height of the ball, e) the time it takes for the ball to reach $y = 100\text{ (ft)}$, and f) the quadratic function $y = f(x)$ that describes the path of the ball.

Solution:

a) When $y = 50\text{ (ft)}$, the second of equations (1) can be rewritten as the following quadratic equation.

$$\boxed{16.1t^2 - 73.54t + 50 = 0} \tag{2}$$

Besides using our calculator directly, there are ***three*** basic ***analytical methods*** for finding the ***roots*** of any quadratic equation – the ***quadratic formula***, ***completing the square***, and ***factoring***.

Method 1: Using the ***quadratic formula***, $\boxed{t_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$, we find the roots to be

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)50}}{2(16.1)} = 2.2839 \pm 1.4527 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.8312 \approx 0.831\text{ (s)} \\ 3.7366 \approx 3.74\text{ (s)} \end{cases}}$$

Method 2: To ***complete the square*** of the quadratic equation, we first divide through by the coefficient of t^2 to get, $\boxed{t^2 - 4.5677t + 3.1056 = 0}$. Then, we complete the square as follows by ***adding*** and ***subtracting*** the term $\left(\dfrac{4.5677}{2}\right)^2$ to the equation.

$$\begin{aligned} t^2 - 4.5677t + 3.1056 &= t^2 - 4.5677t + \left(\frac{4.5677}{2}\right)^2 + 3.1056 - \left(\frac{4.5677}{2}\right)^2 \\ &= \left(t - \frac{4.5677}{2}\right)^2 + 3.1056 - \left(\frac{4.5677}{2}\right)^2 \\ &= (t - 2.2838)^2 - 2.1101 \end{aligned}$$

or

$$(t - 2.2838)^2 = 2.1101 \quad \Rightarrow \quad t - 2.2838 = \pm\sqrt{2.1101} \quad \Rightarrow \quad \boxed{t = 2.2838 \pm 1.4526}$$

$$\Rightarrow \boxed{t_{1,2} \approx \begin{cases} 0.8312 \approx 0.831 \text{ (s)} \\ 3.7364 \approx 3.74 \text{ (s)} \end{cases}}$$

Method 3: The final method is ***factoring***. Factoring is useful if the roots of the quadratic are ***integers***; however, in this case, as in many real engineering problems, they are not. Knowing the roots from either of the above methods, we recognize that the factored form of equation (2) can be written as follows.

$$\boxed{\begin{aligned} (t - 0.8312)(t - 3.7364) &= t^2 - (0.8312 + 3.7364)t + (0.8312 \times 3.7364) \\ &\approx t^2 - 4.5677t + 3.1056 \end{aligned}}$$

b) The ball ***hits the green*** when $y = 3$ (ft), so we can write $\boxed{16.1t^2 - 73.54t + 3 = 0}$. Using the quadratic formula,

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)3}}{2(16.1)} = 2.2839 \pm 2.2427 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.0411 \text{ (s)} \\ 4.5265 \approx 4.53 \text{ (s)} \end{cases}}$$

Clearly, the second of these two times is the answer for which we are looking.

c) We can find out how close the ball lands to the pin by calculating the *x*-coordinate of the ball when it hits.

$$\boxed{x(t)\big|_{t=4.5265} = 61.71(4.5265) = 279.33 \approx 279 \text{ (ft)}} \tag{3}$$

So, the ball ***lands*** about 21 (ft) ***short of the pin***.

d) The ***maximum height*** of the ball occurs at the ***midpoint*** of the times found in parts (a) and (b) for the ball to be at heights of 50 (ft) and 3 (ft). The analysis of those parts gives this time to be $t = 2.2839\ \text{(s)}$.

$$\boxed{y_{\max} = y(t)\big|_{t=2.2839} = \left(73.54\times 2.2839\right) - \left(16.1\times 2.2839^2\right) \approx 83.977 \approx 84\ \text{(ft)}}$$

e) How much time does it take the ball to reach a height of $y = 100\ \text{(ft)}$? Given what we already know from part (d), we expect that there are ***no times*** when $y = 100\ \text{(ft)}$. Using the quadratic formula

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)100}}{2(16.1)} = \boxed{2.2839 \pm 0.9976\, i} \quad \Rightarrow \quad \text{complex roots}$$

These are both ***complex roots***, so ***no real roots exist***.

f) The quadratic function $y(x)$ can be found by ***solving*** the first of equations (1) for t and ***substituting*** that expression into the second of equations (1) as follows.

$$\boxed{y(x) = 73.54\left(\frac{x}{61.71}\right) - 16.1\left(\frac{x}{61.71}\right)^2 = 1.1917\, x - \left(4.2278\times 10^{-3}\right)x^2}$$

With this function, we can answer questions about the ball's position without involving the time t. For example, we can ask the question: At what x-values will $y = 3\ \text{(ft)}$? Again, using the quadratic formula, for $\boxed{\left(4.2278\times 10^{-3}\right)x^2 - 1.1917\, x + 3 = 0}$ we find

$$x_{1,2} = \frac{1.1917 \pm \sqrt{1.1917^2 - (4\times 4.2278\times 10^{-3}\times 3)}}{2\times 4.2278\times 10^{-3}} = 140.94 \pm 138.40 = \begin{cases} 2.54\ \text{(ft)} \\ 279.3\ \text{(ft)} \end{cases}$$

The second result is the same as we found in equation (3).

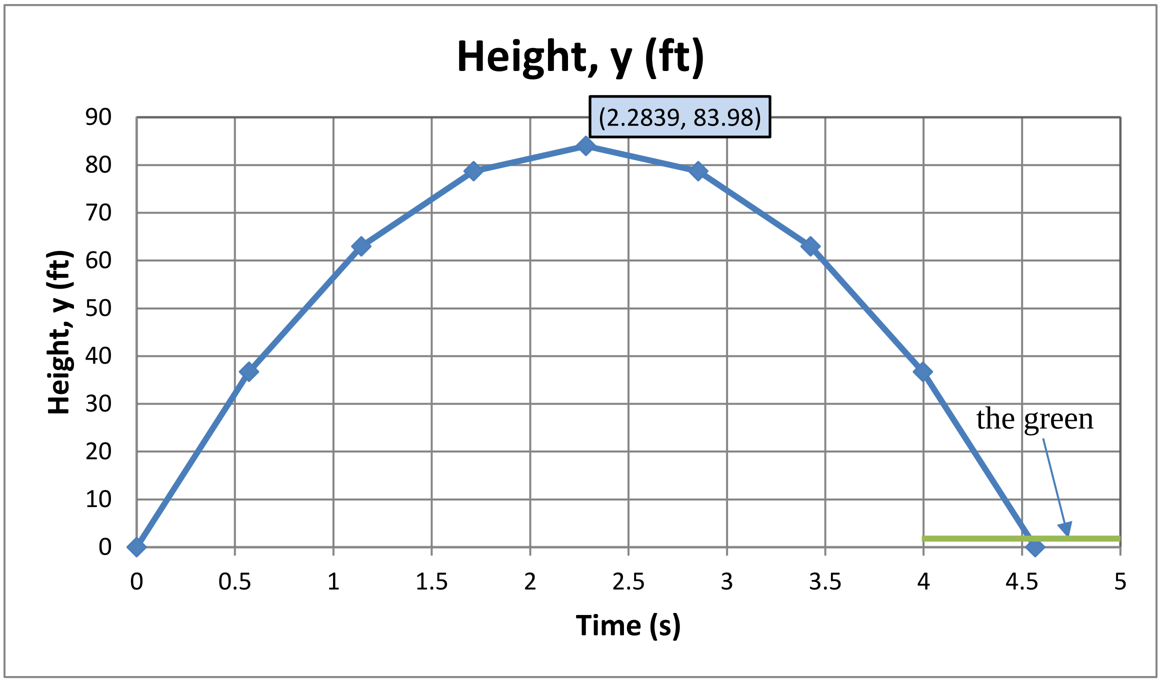


Figure 1. Height of Ball as a Function of Time

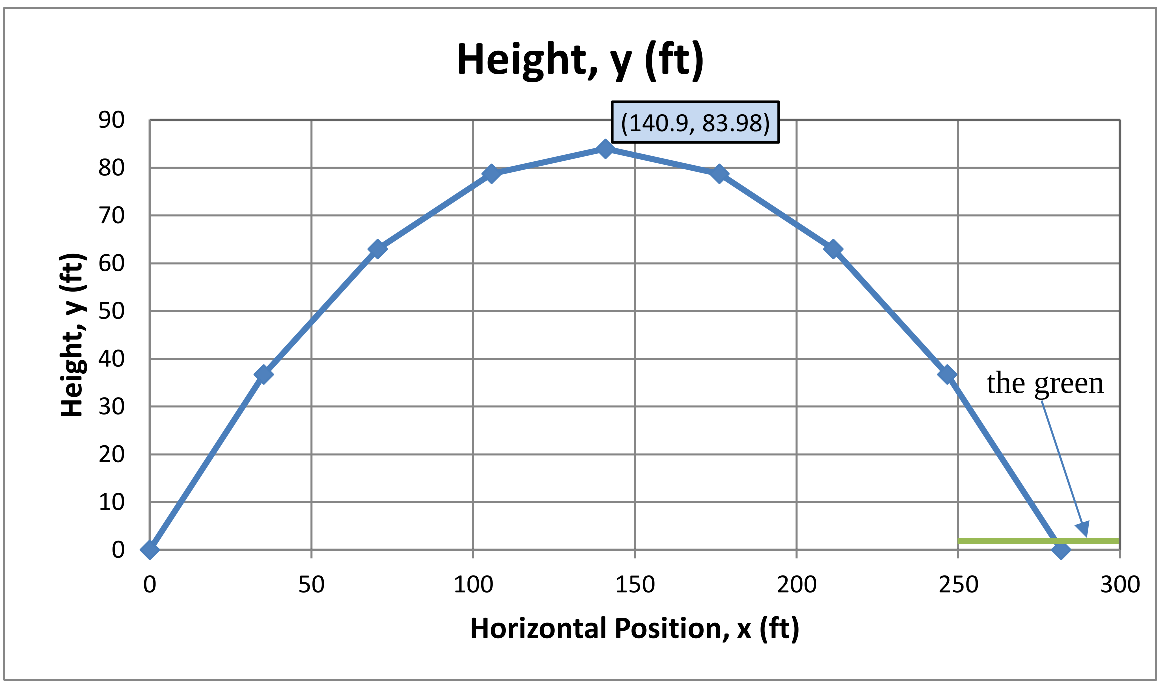


Figure 2. Height of Ball as a Function of Horizontal Position