

Elementary Engineering Mathematics

Application of Trigonometric Functions in Mechanical Engineering: Part I

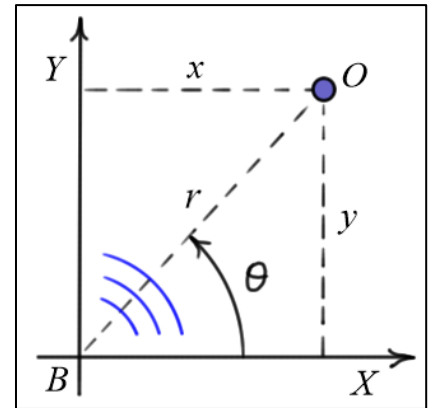
Given: The **position** of an object O is to be found relative to the base B . The distance r and the angle θ were found using radar.

Find: Find the X and Y coordinates of the object O .

Solution: The distances x and y are related to r and θ by the **trigonometric functions** defined for a **right triangle**. In particular,

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

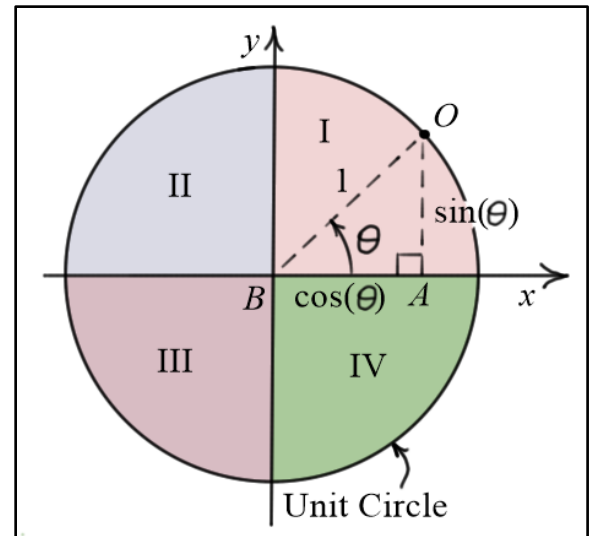


Note: The distance r and angle θ are called the **polar coordinates** of O , and the distances x and y are called the **Cartesian coordinates** of O .

Sine and Cosine Functions:

These trigonometric functions can be defined based on lengths found within the **unit circle** (a circle of radius one). The triangle ABO is a right triangle with hypotenuse length $r=1$ and sides of length $x=\cos(\theta)$ and $y=\sin(\theta)$. Using the **Pythagorean theorem**, we see that the $\sin(\theta)$ and $\cos(\theta)$ are related by the expression

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad (2)$$



Note also that due to the directions of the X and Y axes, these functions **change sign** as θ is varied from $0 \rightarrow \pm 360$ degrees (or $0 \rightarrow \pm 2\pi$ radians). Table 1 summarizes how the $\sin(\theta)$ and $\cos(\theta)$ functions change as θ is varied through each of the **four quadrants** (labeled as I, II, III, and IV in the diagram). Table 2 provides the values for some commonly used angles. A more detailed plot of the functions is shown below in Figure 1.

Quadrant	I	II	III	IV
$\sin(\theta)$	“+” (0 → +1)	“+” (+1 → 0)	“-” (0 → -1)	“-” (-1 → 0)
$\cos(\theta)$	“+” (+1 → 0)	“-” (0 → -1)	“-” (-1 → 0)	“+” (0 → +1)

Table 1. Variation of Sine and Cosine Functions Through the Quadrants

Angle, θ	0	30 (deg) ($\pi/6$ (rad))	45 (deg) ($\pi/4$ (rad))	60 (deg) ($\pi/3$ (rad))	90 (deg) ($\pi/2$ (rad))
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Table 2. Values for Commonly Used Angles

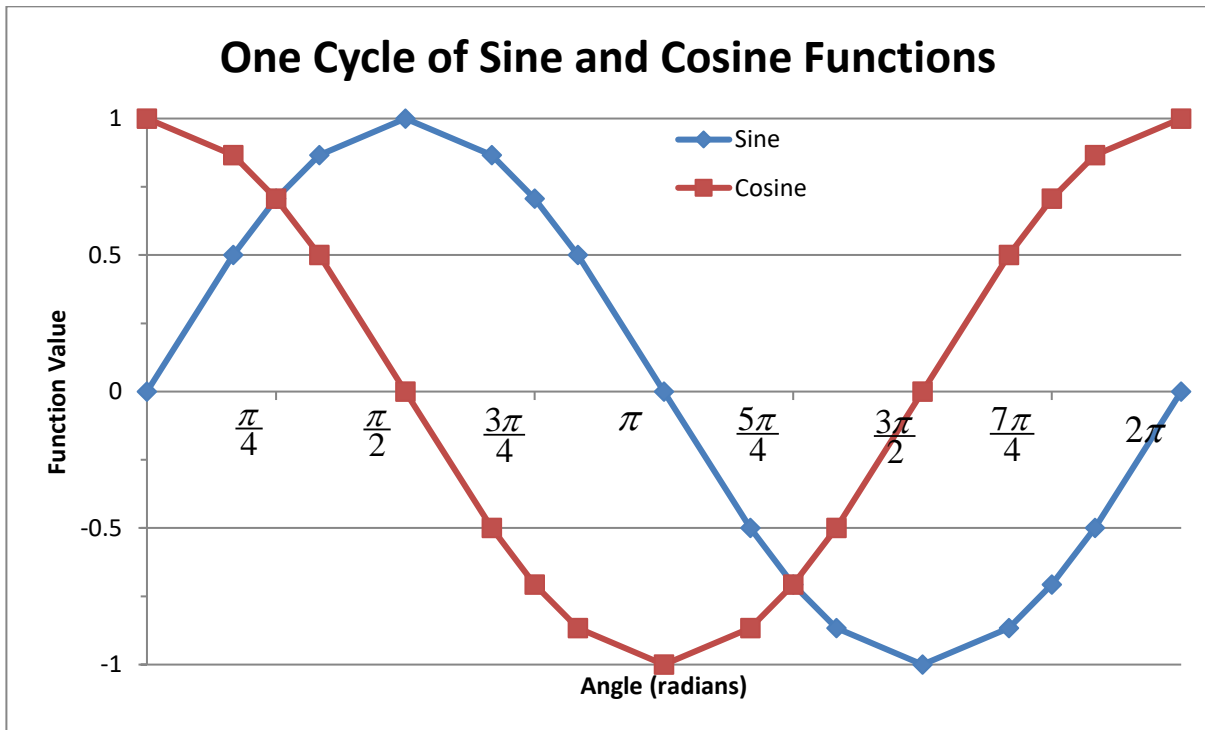


Figure 1. Plot of Sine and Cosine Functions

Tangent Function:

Returning to object O in the first diagram, the distances x and y can be *related directly* using the *tangent* function

$$r = \frac{x}{\cos(\theta)} = \frac{y}{\sin(\theta)} \Rightarrow \boxed{\frac{y}{x} = \frac{\text{opp}}{\text{adj}} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)} \quad \text{or} \quad \boxed{y = x \tan(\theta)} \quad (3)$$

Because the $\tan(\theta)$ is a *ratio* of $\sin(\theta)$ and $\cos(\theta)$, the *sign* of the tangent function is again determined by the *quadrant* of the angle. The results are summarized in Table 3. Note that $\tan(\theta)$

is *undefined* when $\cos(\theta) = 0$, that is when $\theta = \begin{cases} \pi / 2 \text{ (rad) (90 deg)} \\ 3\pi / 2 \text{ (rad) (270 deg)} \end{cases}$

Quadrant	I	II	III	IV
$\sin(\theta)$	“+” (0 → +1)	“+” (+1 → 0)	“-” (0 → -1)	“-” (-1 → 0)
$\cos(\theta)$	“+” (+1 → 0)	“-” (0 → -1)	“-” (-1 → 0)	“+” (0 → +1)
$\tan(\theta)$	“+” (0 → +∞)	“-” (-∞ → 0)	“+” (0 → +∞)	“-” (-∞ → 0)

Table 3. Variation of Sine, Cosine and Tangent Functions Through the Quadrants

Other Trigonometric Functions:

It is also common to define the reciprocals of the sine, cosine, and tangent functions. These are the *cosecant*, *secant*, and *cotangent* functions, respectively.

$$\boxed{\csc(\theta) = \frac{1}{\sin(\theta)}} \quad \boxed{\sec(\theta) = \frac{1}{\cos(\theta)}} \quad \boxed{\cot(\theta) = \frac{1}{\tan(\theta)}} \quad (4)$$

Example 1:

Given: The polar coordinates of an object O are $r = 3500$ (ft) and $\theta = 150$ (deg).

Find: The Cartesian coordinates x and y of O using a) a calculator, and b) the values listed above for commonly used angles

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = r \cos(\theta) = 3500 \times \cos(150) \approx -3031 \text{ (ft)}}$$

$$\boxed{y = r \sin(\theta) = 3500 \times \sin(150) = 1750 \text{ (ft)}}$$

b) Using the values for commonly used angles: Note first that $150 = 180 - 30$ (deg), so

$$\sin(150) = \sin(30) = \frac{1}{2} \quad \text{and} \quad \cos(150) = -\cos(30) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow x = r \cos(\theta) = 3500 \times \frac{-\sqrt{3}}{2} \approx -3031 \text{ (ft)} \quad \text{and} \quad y = r \sin(\theta) = 3500 \times \frac{1}{2} = 1750 \text{ (ft)}$$

Inverse Trigonometric Functions:

Given values for the distances x , y , and/or r , the angle θ can be found using *inverse trigonometric functions* as follows:

$$\theta = \sin^{-1}(y/r) = \cos^{-1}(x/r) = \tan^{-1}(y/x)$$

Example 2:

Given: The Cartesian coordinates of an object O are $x = -3250$ (ft) and $y = 1250$ (ft).

Find: a) the distance r , and b) the angle θ .

Solution:

a) Distance r is found by using the Pythagorean theorem:

$$r = \sqrt{(-3250)^2 + 1250^2} \approx 3482 \text{ (ft)}$$

b) The angle can be found using the inverse tangent function, being careful to *identify* the *correct quadrant* based on the *algebraic signs* of x and y .

$$\theta = \tan^{-1}\left(\frac{+1250}{-3250}\right) \approx 159 \text{ (deg)} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{+1250}{-3250}\right) \approx 2.77 \text{ (rad)}$$

Your *calculator* will probably always give you an angle that is between $-\pi/2$ and $-\pi/2$ radians, so you will have to *adjust* the result by adding π (rad) or 180 (deg). Some may have an “atan2” function that determines the quadrant by individually considering the sign of the numerator and denominator of the ratio.